Theory of Real functions

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Thm 4.2.4 Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, c a cluster point of A. Let $b \in \mathbb{R}$. Then

(a) if $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$ Then

$$\lim_{x \to c} (f+g) = L+M, \quad \lim_{x \to c} (f-g) = L-M, \quad \lim_{x \to c} fg = LM,$$

$$\lim_{x \to c} bf = b \lim_{x \to c} f$$

(b) If $h : A \to \mathbb{R}$, $h(x) \neq 0 \ \forall x \in A$, $\lim_{x \to c} h(x) = H \neq 0$

Then $$\lim_{x \to c} \frac{f}{h} = \frac{L}{H}.$$

Proof. To Sh $\lim_{x \to c} f+g = L+M$.

For this, let (x_n) be a sequence in A, $x_n \neq c$, s.t. $x_n \to c$. $\therefore$ By Thm 4.1.8, $\lim_{n \to \infty} f(x_n) = L$, $\lim_{n \to \infty} g(x_n) = M$. Hence $\lim_{n \to \infty} f(x_n) + g(x_n) = L+M$

$\Rightarrow \lim_{n \to \infty} f+g(x_n) = L+M$ $\therefore$ By Thm 4.1.8,

$\Rightarrow \lim_{x \to c} f+g = L+M$.

$\|^{ly}$ the other parts follows.

$\longleftrightarrow$

Note) If $\lim_{x \to c} f_k(x) = L_k$

Then $\lim_{x \to c} f_1 + f_2 + \cdots + f_n(x) = L_1 + L_2 + \cdots + L_n$.

Also, if $\lim_{x \to c} f = L$ then $\lim_{x \to c} f^n = L^n$.

Example 4.2.5 (a) $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ ~~Provided $\lim_{x \to c} x$ =~~

By Last Thm, $\lim_{x \to c} \frac{1}{x} = \frac{\lim_{x \to c} 1}{\lim_{x \to c} x} = \frac{1}{c}$. $c \neq 0$.

(b) $\lim_{x \to 2} (x^2+1)(x^3-4) = \lim_{x \to 2} (x^2+1) \lim_{x \to 2} (x^3-4)$

$= \left(\lim_{x \to 2} x^2 + \lim_{x \to 2} 1\right)\left(\lim_{x \to 2} x^3 - \lim_{x \to 2} \right.$

$= (4+1)(8-4)$

$= 5 \cdot 4 = 20$.

(c) $\lim_{x \to 2} \frac{x^3-4}{x^2+1} = 4/5$.

(d) $\lim_{x \to 2} \dfrac{x^2-4}{3x-6} = 4/3$

Note that $\lim_{x\to 2} 3x-6 = 0$

$\therefore \lim_{x \to 2} \dfrac{x^2-4}{3x-6} \neq \dfrac{\lim_{x\to 2} x^2-4}{\lim_{x\to 2} 3x-6}$

Consider, $\lim_{x\to 2} \dfrac{x^2-4}{3x-6} = \lim_{x\to 2} \dfrac{\cancel{(x-2)}(x+2)}{3\cancel{(x-2)}} = \lim_{x\to 2} \dfrac{x+2}{3}$

$= 4/3$.

(e) $\lim_{x\to 0} \dfrac{1}{x}$ does not exist in $\mathbb{R}$.

Since $\lim_{x\to c} x = 0$ $\therefore$ We can't use $\lim_{x\to 0} \dfrac{1}{x} = \dfrac{1}{\lim_{x\to 0} x}$.

let $f(x) := 1/x$ then f is ~~unbdd on $\mathbb{R}$~~

not bdd on a neighborhood of 0. $\therefore$ By Thm

4.2.2, it follows that $\lim_{x\to 0} f(x)$ does not exist.

(f) Let p be a polynomial function, then

$\lim_{x\to c} p(x) = p(c)$.

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \text{—} + a_1 x + a_0 \ \forall x \in \mathbb{R}$.

$$\Rightarrow \lim_{x\to c} p(x) = \lim_{x\to c} (a_n x^n + a_{n-1}x^{n-1} + \text{---} + a_1 x + a_0)$$

$$= \lim_{x\to c} a_n x^n + \text{———} + \lim_{x\to c} a_1 x + a_0$$

$$= a_n c^n + a_{n-1} c^{n-1} + \text{---} + a_1 c + a_0 .$$

$$= p(c).$$

(9) One can show that if p & q are two polynomials on $\mathbb{R}$. If $q(c) \neq 0$ then

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} .$$

<u>Thm 4.2.6</u> Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, c a cluster point of A. If $a \le f(x) \le b$ $\forall x \in A$, $x \neq c$. & if $\lim_{x\to c} f$ exists then $a \le \lim_{x\to c} f \le b$.

<u>Proof.</u> Let $L = \lim_{x\to c} f(x)$.

and (x_n) be a sequence in A, $x_n \neq c$, $x_n \to c$, then $\lim_{n\to\infty} f(x_n) = L$.

Since $a \le f(x) \le b$ $\forall x \in A$ with $x \neq c$

$\Rightarrow$ $a \le f(x_n) \le b$

$\Rightarrow$ $a \le \lim_{n\to\infty} f(x_n) \le b$ $\Rightarrow$ $a \le L \le b$.

Theorem (Squeeze Thm) Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$, c a cluster point of A. If

$$f(x) \le g(x) \le h(x) \quad \forall x \in A, x \neq c$$

and if $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$ then $\lim_{x \to c} g(x) = L$.

Example 4.2.8 (a) $\lim_{x \to 0} x^{3/2} = 0 \quad (x > 0)$

Let $f(x) = x^{3/2}$, $x > 0$, we know

$$x < x^{1/2} \le 1 \quad \forall \; 0 < x \le 1$$

$$\Rightarrow \quad x^2 \le x^{3/2} \le x \quad \forall \; 0 < x \le 1$$

$$\Rightarrow \quad x^2 < f(x) \le x \quad \forall 0 < x \le 1$$

$\Rightarrow$ By Squeeze Thm, $\lim_{x \to 0} f(x) = 0$

(b) $\lim_{x \to 0} \sin x = 0$

We know $-x \le \sin x \le x \quad \forall x \ge 0$

$\therefore$ By Squeeze Thm $\lim_{x \to 0} \sin x = 0$ $\left(\because \lim_{x \to 0} x = 0 = \lim_{x \to 0} -x \right)$

(c) $\lim_{x \to 0} \cos x = 1$

We know that

$$\cos x \leq 1 \quad \forall x \in \mathbb{R}$$

Also, $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \text{---}$

$$\Rightarrow 1 - \frac{x^2}{2} \leq \cos x \leq 1 \quad \forall x \in \mathbb{R} \quad \text{— (1)}$$

Since $\lim_{x \to 0} 1 - \frac{x^2}{2} = 1$ $\therefore$ By Squeeze Thm,

$$\lim_{x \to 0} \cos x = 1.$$

(d) $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

By (1) above, $1 - \frac{x^2}{2} \leq \cos x \leq 1$

$$\Rightarrow -\frac{x^2}{2} \leq \cos x - 1 \leq 0$$

$$\Rightarrow -x/2 \leq \frac{\cos x - 1}{x} \leq 0 \quad \forall x > 0$$

and $0 \leq \frac{\cos x - 1}{x} \leq -x/2 \quad \forall x < 0$

$\Rightarrow$ In case $x > 0$, $\lim_{x \to 0} -x/2 = 0$ $\therefore$ By Squeeze Thm, $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

for $x < 0$, $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

(e) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

For $x \geq 0$, $x - \frac{x^3}{\lfloor 3} \leq \sin x \leq x$

For $x \leq 0$, $x \leq \sin x \leq x - \frac{x^3}{6}$

$\Rightarrow$ $1 - \frac{x^2}{\lfloor 3} \leq \frac{\sin x}{x} \leq 1 \quad \forall x \geq 0,$

$1 \leq \frac{\sin x}{x} \leq 1 - \frac{x^2}{6} \quad \forall x \leq 0$

$\Rightarrow$ Since $\lim_{x \to 0} \frac{x^2}{6} = 0$

$\therefore$ $\lim_{x \to 0} 1 - \frac{x^2}{6} = 1$

Hence $\lim_{x \to 0} \frac{\sin x}{x} = 1$ by squeeze Thm.

(f) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

We know $|\sin z| \leq 1 \quad \forall z \in \mathbb{R}$

$\Rightarrow$ $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$\Rightarrow$ $-x \leq x \sin\left(\frac{1}{x}\right) \leq x$

Since $\lim_{x \to 0} |x| = 0$

$$\Rightarrow \lim_{x \to 0} x \sin\left(\frac{1}{5x}\right) = 0.$$

Thm 4.2.9 Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, c a cluster point of A. If $\lim_{x \to c} f > 0$, then $\exists$ a neighborhood $V_\delta(c)$ of c $\ni$ $f(x) > 0$. $\forall\ x \in A \cap V_\delta(c)$, $x \neq c$.

Proof. Let $L = \lim_{x \to c} f(x) > 0$

let $\epsilon = L/2 > 0$ $\quad \exists\ \delta > 0 \ \ni$

$0 < |x - c| < \delta \Rightarrow |f(x) - L| < L/2$

$\Rightarrow L/2 < f(x) < 3L/2$

$\Rightarrow f(x) > L/2 > 0 \quad \forall x \in (c-\delta, c+\delta)$

$\Rightarrow f(x) > 0 \quad \forall x \in V_\delta(c) \cap A, x \neq c.$