Mathematical finance (Sem-V)

When expenditures and receipts are denominated in cash, the net receipts at any time period are termed cash flow, and the series of flows over several periods is termed as a cash flow stream.

* Investment objective is that of tailoring this cash flow stream to be more desireable than it would be otherwise.

* An investment is defined in terms of its resulting cash flow sequence, namely, the amounts of money that will flow to and from an investor over time.

* Comparison Principle :- We evaluate the investment by comparing it with other investments available in the financial market. The financial market provides a basis for comparison.

* Arbitrage :- The simultaneous buying and selling of securities, currency, or commodities in different markets or in its derivative forms in order to take advantage of different prices for the same asset.

earning money without investing anything.

Risk aversion:- It is the behaviour of investors & consumers, who, when exposed to uncertainity, attempt to lower that uncertainty.

Hedging:- It is the process of reducing the financial risks that either arise in the course of normal business operations or are associated with investments.

One form of hedging is insurance where, by paying a fixed amount (a premium), you can protect yourself against certain specified possible losses - such as losses due to fire, theft, or even adverse price changes - by arranging to be paid compensation for the losses you incur.

Pure investment:- It refers to the objective of obtaining increased future return for present allocation of capital.

Ch-2

Interest is frequently called the time value of money.

Simple interest

$$V = (1+rn)A$$

The account grows linearly with time.

Compound interest :- $A = P(1+r)^t$ compounded annually.

otherwise $A = P\left(1+\frac{r}{n}\right)^{(nt)}$.

The seven - ten rule :- Money invested at 7% per year doubles in approximately 10 years. Also, money invested at 10% per year doubles in approx. 7 years.

(Exact at 7% increases by 1.97
at 10% — 1.95)

Compounding at various intervals :-

The effect of compounding on yearly growth is highlighted by stating an effective interest rate, which is equivalent yearly interest rate that would produce the same result after 1 year without compounding.

eg annual rate - 8% compounded quarterly,

$$(1.02)^4 = 1.0824$$

hence effective interest rate is 8.24%.

The Basic yearly rate (8%) is termed as the nomial rate.

Continuous Compounding :- We can imagine dividing the years into smaller and smaller periods, and thereby apply compounding monthly, weekly, daily, or even every minute or secon

This leads to the idea of continuous compounding

We can determine the effect of continuous compounding by considering the limit of ordinary compounding as the number in of periods in a year goes to infinity.

i.e. $$\lim_{m \to \infty} \left[1 + \left(\frac{r}{m}\right)\right]^m = e^r.$$

where $e = 2.7818\ldots$

The effective rate of interest r' is the value satisfying $1 + r' = e^r$.

If the nominal interest rate is 8% per year, then by continuous compounding the growth would be $e^{0.08} = 1.0833$, and hence effective interest rate is 8.33%.

at 10%
1 doubles in 7 years
8 times in 20 years

~~Discount~~

Present value :- It is the current value of a future sum of money or stream of cash flows given a specified rate of return.

In general, \$1 to be received a year in the future has a present value of \$ $\frac{1}{1+r}$, where r is the interest rate.

The process of evaluating future obligations as an equivalent present value is referred as discounting.

The present value of a future monetary amount is less than the face value of that amount. The factor by which the future value must be discounted is called the discount factor.

1-year discount factor is $d_1 = \frac{1}{1+r}$

present value is the discounted amount $d_1 A$.

$$d_k = \frac{1}{\left(1+ r/m\right)^k}$$

m - periods each year   k - Amount at k^{th} period.

Ideal banks :- An ideal bank applies the same rate of interest to both deposits and loans, and it has no service charges or transactions fees. Its interest rate applies equally to any size of principal.

If an ideal bank has an interest rate that is independent of the length of time for which it applies, and that interest is compounded according to normal rules, it is said to be a <u>constant ideal bank</u>.

<u>Future Value (of cash flow streams)</u>

Let we have a fixed time cycle for compounding and let a period be the length of this cycle. We assume that cash flows occur at the end of each period (although some flows might be zero). We take each cash flow and deposit it in a constant ideal bank as it arrives. Then the final balance in our account can be found by combining the results of individual flows.

Let, consider the cash flow stream $(x_0, x_1, \ldots, x_n)$

At the end of n periods the initial cash flow x_0 will have grown to $x_1 (1+r)^n$, where r is the interest rate per period.

The next flow x_1, received after the first period, will at the final time have been in account for only $n-1$ periods, and hence it will have a value of $x_1 (1+r)^{n-1}$.

Likewise x_2 have value $x_2 (1+r)^{n-2}$..

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and x_n from value x_n.

The total value at the end of n periods is therefore $FV = x_0(1+r)^n + x_1(1+r)^{n-1} + \dots + x_n$

⇒ <u>Future value of a stream</u> :- Given a cash flow stream $(x_0, x_1, \dots, x_n)$ and interest rate r each period, the future value of the stream is

$$FV = x_0(1+r)^n + x_1(1+r)^{n-1} + \dots + x_n .$$

<u>Ex</u>!- Consider the cash flow stream $(-2, 1, 1, 1)$ when the periods are years and the interest rate is 10%.

$$FV = -2 \times (1.1)^3 + 1 \times (1.1)^2 + 1 \times 1.1 + 1$$

$$= -2.662 + 1.21 + 1.1 + 1$$

$$= 0.648$$

<u>Present value</u> (of a cash stream).

Consider the stream $(x_0, x_1, \dots, x_n)$. The present value of the first ~~stream~~ element x_0 is just x_0. The present value of x_1 is $\frac{x_1}{1+r}$, because that flow must be discounted by one period. Continuing this way, the present value of the entire stream is

$$PV = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \dots + \frac{x_n}{(1+r)^n}$$

⇒ Present value of a stream:- Given a cash flow $(x_0, x_1, \ldots, x_n)$ and an interest rate r per period, the present value of this cash flow stream is

$$PV = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

Ex:- Consider the cash flow stream $(-2, 1, 1, 1)$. Using an interest rate of 10%, we have

$$PV = -2 + \frac{1}{1.1} + \frac{1}{(1.1)^2} + \frac{1}{(1.1)^3} = 0.487.$$

* The present value of a cash flow stream can be regarded as the present payment amount that is equivalent to the entire ~~system~~ stream. Thus we can think of the entire stream as being replaced by a single flow at initial time.
* Future value is the amount of future payment that is equivalent to the entire system. We can think of the stream as being transformed into single cash flow at period n. The present value of this single equivalent flow is obtained by discounting it by $(1+r)^n$. Thus

$$PV = \frac{FV}{(1+r)^n}$$

In example 4, cash flow stream is $(-2, 1, 1, 1)$

then $PV = 0.487$ & $FV = 0.648$

$$0.487 = PV = \frac{FV}{(1.1)^3} = \frac{0.648}{1.331} = 0.487.$$

<u>Frequent and Continuous Compounding</u>

Suppose that r is the nominal annual interest rate and interest is compounded at m equally spaced periods per year. Suppose that cash flows occur initially and at the end of each period for a total of n periods, forming a stream $(x_0, x_1, \ldots, x_n)$. Then

$$FV = \sum_{k=0}^{n} x_k \left(1 + r/m\right)^{n-k}$$

and

$$PV = \sum_{k=0}^{n} \frac{x_k}{\left(1 + r/m\right)^k}$$

Now, Suppose that the nominal interest rate r is compounded continuously and cash flows occur at times $t_0, t_1, \ldots, t_n$. We denote the cash flow at time t_k by $x(t_k)$. then

$$PV = \sum_{k=0}^{n} x(t_k) e^{-rt_k} \quad \text{and} \quad FV = \sum_{k=0}^{n} x(t_k) \cdot e^{r\, t_{(n-k)}}$$