Planar wave guide:

Maxwell's equation for an isotropic, linear non conducting and nonmagnetic medium.

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad -(1), \qquad \nabla \times H = \frac{\partial D}{\partial t} \quad -(2)$$

$$\nabla \cdot D = 0 \quad -(3) \qquad \nabla \cdot B = 0 \quad -(4).$$

constitutive equations are $B = \mu_0 H$, -(5)

$D = \epsilon E = \epsilon_0 n^2 E$ — (6)

E, D, B, H represents the electric field, electric displacement, magnetic induction and magnetic intensity respectively. μ_0 — free space permeability.

$\epsilon = \epsilon_0 n^2$ represents the dielectric permittivity of the medium, and n, the refractive index.

For such a medium Maxwell's equation can be rewritten as,

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t}, \quad -(7) \qquad \nabla \times H = \epsilon_0 n^2 \frac{\partial E}{\partial t}. \quad -(8)$$

$$\nabla \cdot \epsilon_0 n^2 E = 0 \quad -(9) \qquad \nabla \cdot \mu_0 H = 0 \quad -(10).$$

Wave equation: Take curl of (7)

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\mu_0 \frac{\partial H}{\partial t}\right).$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \frac{\partial}{\partial t}(\nabla \times H) \qquad \text{sub } (\nabla \times H) \text{ from eq } (8)$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \frac{\partial}{\partial t}\left[\epsilon_0 n^2 \frac{\partial E}{\partial t}\right]$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu_0 \epsilon_0 n^2 \frac{\partial^2 E}{\partial t^2} \quad — (9)$$

From ~~consider~~ eq. 3 and eq. 6

$$\text{div}\cdot D = 0$$

$$0 = \text{div}[\epsilon_0 n^2 E]$$

consider the identity

$$\nabla\cdot(U\cdot A) = U(\nabla\cdot A) + A\cdot(\nabla\cdot U)$$

$$U = n^2 \quad A = E,$$

$$0 = n^2(\nabla\cdot E) + E\cdot(\nabla\cdot n^2)$$

$$\nabla\cdot E = -\frac{1}{n^2}[\nabla n^2 E] \quad \text{--- (10)}$$

sub. in eq. (9).

$$\nabla\left(-\frac{1}{n^2}[\nabla n^2\cdot E]\right) - \nabla^2 E = -\mu_0\epsilon_0 n^2\frac{\partial^2 E}{\partial t^2}$$

$$= \nabla^2 E + \nabla\left(\frac{1}{n^2}\nabla n^2\cdot E\right) - \mu_0\epsilon_0 n^2\frac{\partial^2 E}{\partial t^2} = 0 \text{ --- (11)-(a)}.$$

Similarly for magnetic field vector.

$$\nabla^2 H + \nabla\left[\frac{1}{n^2}\nabla n^2\cdot H\right] - \mu_0\epsilon_0 n^2\frac{\partial^2 E}{\partial t^2} = 0 \text{ --- 11 (b)}$$

But n varies in transverse direction

$$n^2 = n^2(x, y). \quad \text{--- (12)}$$

Solutions of eq. 11 (a) and 11 (b). are

$$E = E(x, y)\,e^{i(\omega t - \beta z)}. \quad \text{--- (13)}$$

$$H = H(x, y)\,e^{i(\omega t - \beta z)} \quad \text{--- (14)}$$

where β is wave propogation constant.

Eq. (13) and (14) defines the modes of the system.

Let us assume $n^2 = n^2(x)$. — (15). depends on the x- coordinate only.

We can rewrite the components of Eqs (13) & (14).

$$E_j = E_j(x)\, e^{i(\omega t - \beta z)} \qquad j = x, y, z. \quad — (16)$$

$$H_j = H_j(x)\, e^{i(\omega t - \beta z)} \qquad j = x, y, z. \quad — (17)$$

$$\frac{\partial}{\partial t} \to i\omega, \quad \frac{\partial}{\partial z} = -j\beta.$$

components of Max. equation

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix} = -\mu_0 i\omega \left[\hat{i} H_x + \hat{j} H_y + \hat{k} H_z \right].$$

x- components,

$$\left[\frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y \right] = -\mu_0 i\omega H_x \qquad x$$

y- component.

$$-\left[\frac{\partial}{\partial x} E_z - \frac{\partial}{\partial z} E_x \right] = -\mu_0 i\omega H_y \qquad — (18)$$

z- component.

$$\left[\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \right] = -\mu_0 i\omega H_z.$$

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{bmatrix} = \epsilon_0 n^2(x) \frac{\partial E}{\partial t} = \epsilon_0 n^2(x)\, i\omega \left[\hat{i} E_x + \hat{j} E_y + \hat{k} E_z \right]$$

x, y and z components are.

$$\hat{i}\left[\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y\right] = \hat{i}\,\epsilon_0 n^2_{(x)}\, i\omega E_x$$

$$-\hat{j}\left[\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial z}H_y\right] = \hat{j}\,\epsilon_0 n^2_{(x)}\, i\omega E_y$$

$$\hat{k}\left[\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right] = \epsilon_0 n^2_{(x)}\, i\omega E_z \quad \text{--- } (19)$$

since the medium is infinite along x, & y directions.

Hence the field components are assumed to be uniform or constant i.e. $\frac{\partial}{\partial y} = 0$, substitute $\frac{\partial}{\partial z} = -i\beta$.

From 18

$$-[-i\beta E_y] = -\mu_0 i\omega H_x \rightarrow \beta E_y = -\mu_0 \omega H_x \quad \text{--- } 20$$

$$-\left[\frac{\partial}{\partial x}E_z - (-i\beta)E_x\right] = -\mu_0 i\omega H_y \quad \text{--- } (21).$$

$$\frac{\partial}{\partial x}E_y = -\mu_0 i\omega H_z \quad \text{--- } (22).$$

From 19

$$i\beta H_y = \epsilon_0 n^2_{(x)}\, i\omega E_x \quad \text{--- } (23).$$

$$-\left[\frac{\partial}{\partial x}H_x + i\beta H_x\right] = \epsilon_0 n^2_{(x)}\, i\omega E_y \quad \text{--- } (24).$$

$$\frac{\partial}{\partial x}H_y = \epsilon_0 n^2_{(x)}\, i\omega E_z \quad \text{--- } 25.$$

Eqs. (20), (22), 24, involves only E_y, H_x and H_z.

Eqs. (21) (23), (25) involves only E_x, H_y and E_z.

So, for such a waveguide configuration, Maxwell's equations reduces to two independent set of equations.

Eqs. 20, 22, 24. are known as TE mode.

21, 23 & 25 are known as TM mode.