Uniform Convergence

In last lecture we have seen pointwise convergence of sequence of functions, in which ~~we are first~~ is ~~taking $x \in A_0$ and then applying~~

Let $\{f_n\}$ be a seq. of function on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$ and let $f : A_0 \to \mathbb{R}$. We say the seq. $\{f_n\}$ converges (pointwise) to f on A_0, if for each $x \in A_0$ and for every $\epsilon > 0$, there exist $K \in \mathbb{N}$ (here K depends on ϵ and also on x) such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall \ n \geq K.$$

We we change x in above defination, the value of K may be changed.

If K does not depend on x, but only on ϵ, we call that convergence uniform convergence.

Def^n (Uniform convergence):- A sequence $\{f_n\}$ of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if for each $\epsilon > 0$, there exists a $K \in \mathbb{N}$ (depending on ϵ only) such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall \ n \geq K$$

and $\forall \ x \in A_0$.

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Note:- from the def^n of pointwise convergence and uniform convergence, we have

uniform convergence implies pointwise convergence

But converse is not true.

Example:- (1) The sequence $f_n(x) = \frac{nx}{1+n^2x^2}, x \in \mathbb{R}$ is pointwise convergent sequence but not uniform convergent.

Sol^n As $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{1/n + nx^2} = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ the seq. is pointwise converges to 0.

Now we will show that the seq. is not uniform convergent.

Let if possible $\{f_n\}$ is uniformly convergent in $\mathbb{R}$ to 0. Therefore by defination,

for every $\epsilon > 0$, $\exists$ $K \in \mathbb{N}$ s.t.

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \epsilon \quad \forall n \geq K \text{ and } \forall x \in \mathbb{R}$$

Take $\epsilon = 1/4$.

Then by def^n, $\exists$ $K' \in \mathbb{N}$ s.t.

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \frac{1}{4} \quad \forall n \geq K' \text{ \& } \forall x \in \mathbb{R}$$

Let $x = \frac{1}{K'}$ and $n = K'$ in particular

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then $\left| \dfrac{k'\left(\frac{1}{k'}\right)}{1+\left(k'\cdot\frac{1}{k'}\right)^2} \right| < \dfrac{1}{4}$

$\Rightarrow \dfrac{1}{2} < \dfrac{1}{4}$ which is contradiction.

$\therefore \{f_n\}$ is not uniformly convergent.

(2) Show the seq. $\{f_n\}$ where $f_n(x) = \dfrac{n}{x+n}$ is uniformly convergent in $[0,a]$ where $a>0$.

Soln → firstly as

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{n}{x+n} = \lim\limits_{n\to\infty} \dfrac{1}{\frac{x}{n}+1} = 1 \quad \forall x \geq 0$

$\Rightarrow f_n(x)$ converges pointwise to $f(x) = 1$ for all $x \geq 0$.

Now we show uniform convergence.

Let $\epsilon > 0$,

Consider

$|f_n(x) - f(x)| = \left| \dfrac{n}{x+n} - 1 \right| = \dfrac{x}{x+n}$

We have to find $k \in \mathbb{N}$ s.t.

$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq k \ \& \ \forall x \in [0,a]$

i.e. $\dfrac{x}{x+n} < \epsilon$

i.e. $\dfrac{x+n}{x} > \dfrac{1}{\epsilon} \Rightarrow n > x\left(\dfrac{1}{\epsilon} - 1\right)$

As $x \leq a$, let k be the natural no. greater than $a\left(\frac{1}{\epsilon} - 1\right)$, then

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$$\left| \frac{n}{x+n} - 1 \right| < \epsilon \quad \forall\, n \geq K \text{ and } \forall\, x \in [0,a]$$

$\therefore$ $f_n(x)$ converges uniformly to $f(x) = 1$ in $[0,a]$.

Uniform norm:

In discussing uniform convergence, it is useful to use the notion of uniform norm on a set of bounded functions.

Def^n:- If $A \subseteq \mathbb{R}$ and $\phi : A \to \mathbb{R}$ is a function, we say ϕ is bounded on A if the set $\phi(A)$ is a bounded subset of $\mathbb{R}$. If ϕ is bounded we define uniform norm of ϕ on A by

$$\|\phi\|_A := \sup\{ |\phi(x)| : x \in A \}.$$

Example:-

① Let $f(x) = x$ on $[1,3] = A$

then f is bounded on $[1,3]$

and $\|f\|_A = \sup\{ f(x) : x \in A = [1,3] \} = 3$

$\therefore$ $\|f\|_A = 3$.

② Let $f(x) = e^x$ on $A = [0,3)$

then f is bounded on A

and $\|f\|_A = \sup\{ f(x) : x \in [0,3) \} = e^3$.

$\therefore$ $\|f\|_A = e^3$.

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Note:- Let $\epsilon > 0$, such that

$$\|\phi\|_A \leq \epsilon \iff |\phi(x)| < \epsilon \ \forall \ x \in A \quad -(*)$$

Lemma:- A sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $\|f_n - f\|_A \to 0$.

Proof:- Let (f_n) converges uniformly to f on A.

$\Rightarrow$ for any $\epsilon > 0$, $\exists$ $K \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq K, \forall x \in A.$$

Then by $(*)$, we have

$$\|f_n - f\| < \epsilon \quad \forall \ n \geq K.$$

$$\Rightarrow \quad \|f_n - f\| \to 0.$$

Conversely, let $\|f_n - f\| \to 0$.

$\Rightarrow$ for $\epsilon > 0$, $\exists$ $K \in \mathbb{N}$ s.t.

$$\|f_n - f\| < \epsilon \quad \forall \ n \geq K.$$

Then by $(*)$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq K \text{ \& } \forall \ x \in A.$$

$\Rightarrow$ $f_n(x)$ converges uniformly to f on A.

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Examples:-

(1.) Let $f_n(x) = x^n$ for $x \in A := [0,1]$ and

let $f(x) = \begin{cases} 0 & , 0 \le x < 1 \\ 1 & , x = 1 \end{cases}$

then $f_n(x)$ does not converge uniformly to f on A.

Sol^n → As $f_n(x) - f(x) = \begin{cases} x^n & , 0 \le x < 1 \\ x^n - 1 = 0 & , x = 1 \end{cases}$

then

$$\|f_n - f\|_A = \sup \begin{cases} x^n & , 0 \le x < 1 \\ 0 & , x = 1 \end{cases} = 1$$

$\therefore$ $\|f_n - f\|_A = 1$

$\Rightarrow$ $\|f_n - f\|_A \to 1$ but not on 0.

$\therefore$ f_n does not converge uniformly to f on A.
(By lemma).

(2.) Let $f_n(x) = \dfrac{x^2 + nx}{n}$ for $x \in A := [0,8]$ and

let $f(x) = x$. then $f_n(x)$ converges uniformly to f on A.

Sol^n → As $f_n(x) - f(x) = \dfrac{x^2 + nx}{n} - x = \dfrac{x^2}{n}$

then $\|f_n - f\|_A = \sup\{x^2/n : x \in A = [0,8]\} = \dfrac{64}{n}$

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$\therefore \quad \|f_n - f\| = \frac{64}{n} \to 0 \quad as \quad n \to \infty.$

$\therefore \quad \|f_n - f\| \to 0.$

$\Rightarrow \{f_n\}$ converges uniformly to f on A. (By Lemma).

Exercise :-

Q.1 Show that $f_n(x) = \frac{nx}{1+nx}$ converges uniformly on $[a, \infty)$. for $a > 0$.

Q.2. Show that $f_n(x) = \frac{x^n}{1+x^n}$ converges uniformly on $[0, b]$ for $0 < b < 1$, but is not uniform on the interval $[0,1]$.

Series of functions :-

Def^n :- If (f_n) is a sequence of functions defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of partial sums (s_n) of the infinite series $\sum f_n$ is defined for $x \in D$ by

$$s_1(x) := f_1(x)$$
$$s_2(x) := f_1(x) + f_2(x)$$
$$\vdots$$
$$s_n(x) := f_1(x) + f_2(x) + \cdots + f_n(x)$$
$$\vdots$$

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If the sequence (s_n) of functions converges to a function f on D, we say that the infinite series of functions Σf_n converges to f on D.

If the sequence (s_n) of functions converges uniformly to a function f on D, we say that Σf_n is uniformly convergent to f on D.

Tests for uniform convergence :-

Thm (Cauchy Criterion) :- Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series Σf_n is uniformly convergent on D if and only if for every $\epsilon > 0$ there exists an M (depending on ϵ) such that if $m > n \geq M$, then

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon \text{ for all } x \in D.$$

(Imp.)
Thm :- (Weierstrass M-Test) :- Let (M_n) be a seq. of positive real number such that $|f_n(x)| \leq M_n \; \forall x \in D$, $n \in \mathbb{N}$. If the series ΣM_n is convergent, then Σf_n is uniformly convergent on D.

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Proof:- Let $\sum M_n$ is convergent

Then By Cauchy criterion of series of real numbers.

for $\epsilon > 0$, $\exists$ ~~...~~ $K \in \mathbb{N}$ s.t.

$$M_{n+1} + \cdots + M_m < \epsilon \quad \forall \; m > n \geq K. \quad (1)$$

And as $|f_n(x)| \leq M_n \;\; \forall \; x \in D$.

$$\therefore \; |f_{n+1}(x) + \cdots + f_m(x)| \leq M_{n+1} + \cdots + M_m \quad \forall \; x \in D$$

$\therefore$ eq. (1) becomes

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon \quad \forall \; m > n \geq K$$
$$\& \; \forall \; x \in D$$

Then By Cauchy Criterion stated above,

$\sum f_n$ is uniformly convergent on D. ∎

Examples

(1.) Show the uniform convergence of $\sum f_n$, where $f_n(x) = \dfrac{1}{x^2 + n^2}$

Sol^n As $\quad x^2 + n^2 > n^2 \quad \forall \; n, \; \forall \; x \in \mathbb{R}$

$$\Rightarrow \frac{1}{x^2 + n^2} < \frac{1}{n^2} \quad \forall n, \; \forall \; x \in \mathbb{R}$$

$$\Rightarrow f_n(x) < \frac{1}{n^2} \quad \forall n, \; \forall \; x \in \mathbb{R}$$

let $M_n = 1/n^2$

$\therefore$ $f_n(x) \le M_n \quad \forall n \in \mathbb{N} \;\&\; \forall x \in \mathbb{R}$

and as $\sum M_n = \sum 1/n^2$ converges

$\therefore$ By Weierstrass M-test

$\sum f_n$ converges uniformly on $\mathbb{R}$.

(2.) Show uniform convergence of $\sum_{n=0}^{\infty} x^n$ on $[-\alpha, \alpha]$, where $0 < \alpha < 1$.

$\rightarrow$ As $|x| \le \alpha$

$\rightarrow |x^n| \le \alpha^n$

$\rightarrow |f_n(x)| \le \alpha^n \quad \forall n \in \mathbb{N}, \forall x \in [-\alpha, \alpha]$.

let $M_n = \alpha^n$

then $|f_n(x)| \le M_n \quad \forall n \in \mathbb{N}, \forall x \in [-\alpha, \alpha]$.

And as $\sum M_n$ converges whenever $\alpha < 1$.

$\therefore$ By Weierstrass M-test.

$\sum f_n$ converges uniformly on $[-\alpha, \alpha]$.

where $0 < \alpha < 1$.

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Exercise.

Show the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by

(i) $(nx)^{-2}$ on $[1, \infty)$.

(ii) $(n+1)x^n$ on $[-\alpha, \alpha]$ where $0 < \alpha < 1$.

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