Linearity & Superposition Principle:

The state of a system does not have to be represented by a single wavefunction. It can be rep. by superposition of two or more wavefunction.

If $\psi_1(x,t)$ & $\psi_2(x,t)$ separately satisfy the S.E. the wave fun

$$\psi(x,t) = \alpha_1 \psi_1(x,t) + \alpha_2 \psi_2(x,t)$$

also satisfies S.E. where α_1 & α_2 are complex numbers.

STATIONARY STATES:

From the two solution of S.E. (T.D.) & (T.Ind.) we have

$$\psi(x,t) = \psi(x)\, e^{-\frac{iEt}{\hbar}}$$

$$|\psi(x,t)|^2 = |\psi(x)|^2$$

⇒ Probability density is independent of time. Hence $\psi(x,t)$ is time dep. So every exp. value for such wave func. will be constant in time,

So for the states for which $\langle x \rangle$ = constant & $\langle p \rangle = 0$ are known as stationary states (criteria for stationary sol. P-10)

STATES of Definite Total Energy:

As $$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

So S.E. can be written as

$$\hat{H}\psi = E\psi$$

$$\langle H \rangle = \int \psi^* \hat{H} \psi \, dx$$

$$= \int \psi^* E \psi \, dx \quad [\because \hat{H}\psi = E\psi]$$

$$= E \int \psi^* \psi \, dx$$

$\int \psi^* \psi \, dx = 1$ from normalisation condition

So, $\langle H \rangle = E$

Also $\langle H^2 \rangle = E^2$

Variance $\sigma_H^2 = \langle H^2 \rangle - (\langle H \rangle)^2$

$\Rightarrow \sigma_H^2 = 0$

This means that there is no energy spread which means every member of sample has same value of total energy.

Conclusion: A separable solution has property that every measurement of T.E. of sample will return same value of E.

General solution: General soln of eqn

$$\frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E\psi$$

The soln of this eqn is infinite collection of soln for time indep. S.E.

$\psi_1(x), \psi_2(x)$ ---- be soln with dep. const. E_1, E_2 ..... So it means there is diff. wave function for each allowed state.

$$\Psi_1(x,t) = \psi_1(x) e^{-iE_1 t/\hbar}$$

$$\Psi_2(x,t) = \psi_2(x) e^{-iE_2 t/\hbar}$$

So, General soln of Time Independent S.E. is sum of all individual

$$\Psi(x,t) = \sum_{n=1}^{\infty} C_n \psi_n(x)\, e^{-iE_n t/\hbar}$$

For General soln $\psi(x,t)$, probability expectation value depend on time as exp. term will not cancel out as E is different for each Stationary st

What does C_n indicate?

Loosely speaking C_n tells us the amount of ψ_n that is contained in Ψ

$|C_n|^2 \longrightarrow$ Probability of that measurement of the energy would yield the value

Sum of probabilities = 1

$$\therefore \boxed{\sum_{n=1}^{\infty} |C_n|^2 = 1}$$

Expectation value of energy

$$\boxed{\langle H \rangle = \sum_{n=1}^{\infty} |C_n|^2 E_n}$$

This is probability of getting a particular energy is Indep. of time & hence is Indep. of time. This is manifestation of conservation of energy in Quantum Mechanics.

Verification of superposition principle.

If $\psi_1(x)$ & $\psi_2(x)$ be two soln of S.E. then,

$$\frac{d^2\psi_1}{dx^2} + \frac{2m}{\hbar^2}[E-V]\psi_1(x) = 0 \quad (1)$$

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}[E-V]\psi_2(x) = 0 \quad (2)$$

$c_1 \times (1)$ + $c_2 \times (2)$

$$\frac{d^2}{dx^2}[c_1\psi_1 + c_2\psi_2] + \frac{2m}{\hbar^2}[E-V)[c_1\psi_1 + c_2\psi_2] = 0$$

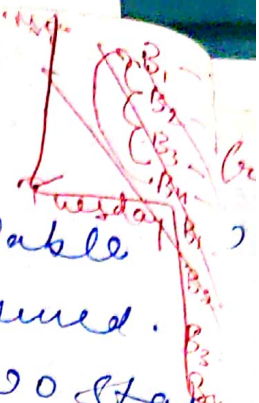
$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E-V)\psi = 0$$

with $\psi = c_1\psi_1 + c_2\psi_2$

So, ψ is a soln of S.E. So principle of linear superposition is obeyed. So two linearly indep. soln ψ_1 & ψ_2 are complete set in the sense that every solution of S.E. can be expressed as linear comb. of ψ_1 & ψ_2.

Mathematically superposition principle means if you linearly combine any number of solutions of an equation, the linear combination is itself a solution. The basic equation of Q.M. is S.E, which is linear d.E. So its soln obey superposition principle.

What actually superposition Principle means?



Suppose a dynamical variable of a system is to be measured. In an expt. Let ψ_1 & ψ_2 be two states such that if an observation is made in state ψ_1, we get definite result λ_1, & if obs. is made in ψ_2 the result is definitely λ_2. What will be the result of the observation if the system is in the superposed state $\psi = c_1\psi_1 + c_2\psi_2$. The result will be sometimes λ_1 & sometime λ_2. No other result will be ever obtained. How it cannot be predicted when we would get λ_1 & when we would get λ_2. We can only say that the prob of getting λ_1 is $|c_1|^2$ & probability of getting λ_2 is $|c_2|^2$

The superposition principle is the that a system is in all possible states at the same time until it is measured. After measurement it falls to one of the eigen states thus, destroying the original state. The superposition principle explains the quantum ordered observed with many experiments.

Criteria for stationary solutions.

(1) Potential should be real & time independ. so that Hamiltonian is time independent- $H \neq H(t)$ enables us to apply the method of Separation of variables.

2) we should find an eigenfunction $\psi(x)$ for Hamiltonian operator $\hat{H}$ with eigenvalue E

$$\hat{H}\psi(x) = E\psi(x)$$

3) Probability density is time independent.

$$\rho \neq \rho(t)$$

4) The time ind. S.E. $\hat{H}\psi(x) = E\psi(x)$ can have infinite number of solutions.

i.e. $\psi_1(x) \cdots \psi_n(x)$

A combination of these solutions of time indep. S.E. (at $t=0$) is

$$\psi(x) = \psi(x,0) = c_1\psi_1 + c_2\psi_2 + \cdots c_n\psi_n$$

$$= \sum c_n \psi_n(x)$$

General sol^n of time dep. S. Equation.

$$\Psi(x,t) = c_1\psi_1(x,t) + \cdots + c_n\psi_n(x,t)$$

$$= \sum c_n \psi_n(x) e^{-iE_n t/\hbar}$$

$$= \sum_n \psi_n(x,0) e^{-iE_n t/\hbar}$$

$|\Psi(x,t)|^2$ = fun of x & t

So, gen. sol^n is not a stationary state sol^n.

so to find the time development of wavefunction one has to start w.r.t time independent eqn & construct linear combination!

$$\psi(x,0) = \sum_n c_n \psi_n(x)$$

Then associate the characteristic time dep. to individual term to get

$$\Psi(x,t) = \sum_n c_n \psi_n(x)\, e^{-iE_n t}$$

which is soln of TDSE. So starting from $\psi(x,0)$ we can find $\Psi(x$

COMMUTATOR:

EIGEN VALUES ARE REAL:

The set of energies & wave functions obtained by solving any QM problem is soln of an eigen value eqn i.e.

$$\hat{H}\psi_n = E_n \psi_n \quad -(1)$$
$$\hat{H}\psi_m = E_m \psi_m \quad -(2)$$

Multiply (1) by ψ_m^* & (2) by ψ_n^* & take diff.

$$\psi_m^* \hat{H} \psi_n - \psi_m \hat{H} \psi_n^* = E_n \psi_n \psi_m^* - E_m \psi_m \psi_n^*$$

Integrating.

$$\int \psi_m^* \hat{H} \psi_n \, d\tau - \int \psi_m \hat{H} \psi_n^* \, d\tau = \int (\psi_n^* \hat{H} \psi_m)^*$$
$$= (E_n - E_m) \int \psi_m^* \psi_n \, d$$

By using Hermitian property

$$(E_n - E_m^*) \int \psi_m^* \psi_n \, d\tau = 0$$

When $n = m$

$$\int \psi_m^* \psi_m = 1 \quad \text{(Normalization)}$$

$$\therefore \quad E_m - E_m^* = 0$$

$$\Rightarrow E_m = E_m^*$$

Hence the energy eigen values should be real because real part is measurable.

When $n \neq m$

2nd factor in eqn will vanish so,

$$\int \psi_m^* \psi_n \, d\tau = 0 \quad \text{if } n \neq m$$

$\Rightarrow$ Eigenfunction belonging to different eigenvalues are orthogonal.

So, we can write ⊙

ORTHONORMALITY CONDITION

$$\int \psi_m^*(x) \, \psi_n(x) \, dx = \delta_{mn} \begin{cases} = 1 \; ; & n = m \\ = 0 \; ; & m \neq n \end{cases}$$

mutually exclusive physical states $\Leftrightarrow$ orthogonality.

$\langle \phi | \psi \rangle = 0$ $\Leftrightarrow$ mutually exclusive physical state. If one of them is true in the sense that it is a correct description of quantum system, the other is false i.e. an incorrect description of system.