Automorphism group of infinite cyclic group

Let G be infinite cyclic group.

i.e $G = \langle a \rangle = \{a^i \mid i \in \mathbb{Z}\}$ be an infinite cyclic group, $a \in G$

Let $T \in Aut(G)$

$T : G \to G$ (T is 1-1, onto, Homo.)

let $a \in G \Rightarrow T(a) \in G = \langle a \rangle$

$\Rightarrow T(a) = a^m$ for some $m \in \mathbb{Z}$

Since T is onto

so, for $a \in G$, $\exists\, x \in G$ s.t $T(x) = a$

As $x \in G$
$\Rightarrow x = a^n$ for some $n \in \mathbb{Z}$

$\Rightarrow a = T(x) = T(a^n)$

$= (T(a))^n$ ($\because$ T is Homo.)

$\left[\begin{aligned} T(a^2) &= T(a \cdot a) \\ &= T(a)T(a) \\ &= (T(a))^2 \end{aligned} \right]$

$= (a^m)^n$

$= a^{mn}$

$\Rightarrow a = a^{mn}$

$\Rightarrow a^{mn-1} = e$

$\Rightarrow mn - 1 = 0$

$\Rightarrow mn = 1$

[Since G is infinite cyclic group so, $a^i = e \iff i = 0$]

As, $m, n \in \mathbb{Z}$

so, $m = +1$ or -1
$n = +1$ or -1 (possible cases)

$\therefore T(a) = a^m$
$= a^1$ or a^{-1}

$\underbrace{T(a) = a}_{\text{Identity map}}$ or $T(a) = a^{-1}$

$\therefore Aut(G) = \{I, T\}$, where $T(a) = a^{-1}$

$O(Aut(G)) = 2$

Every group of prime order is cyclic

$\therefore Aut(G)$ is cyclic, where G is infinite cyclic group.

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then $Aut(G) \cong Z_2$

Example :
let $(Z, +)$; is infinite cyclic group
then $Aut(Z) \cong Z_2$ (by above result)

[Every finite cyclic group of order 'n' is isomorphic to Z_n]

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Characteristic Subgroup

(Def^n) A subgroup H of G is said to be characteristic subgroup if $\phi(H) = H$ $\forall$ $\phi \in Aut(G)$.

Remark: If $\sigma(H) \subset H$ $\forall$ $\sigma \in Aut(G)$ then H is characteristic subgrp of G.

Since $\sigma(H) \subset H$ $\forall$ $\sigma \in Aut(G)$ — (1)

$\Rightarrow$ σ is 1-1, onto, Homo.

$\Rightarrow$ σ^{-1} is also 1-1, onto, Homo.

$\Rightarrow$ $\sigma^{-1} \in Aut(G)$

using condition (1)

$\sigma^{-1}(H) \subset H$ — (2)

$\Rightarrow$ $\sigma\sigma^{-1}(H) \subset \sigma(H)$

$\Rightarrow$ $H \subset \sigma(H) \subset H$ (from (2))

$\Rightarrow$ $H \subset \sigma(H) \subset (H)$

i.e $H \subset \sigma(H) \subset H$

$\therefore$ $\sigma(H) = H$

$\therefore$ H is characteristic subgrp of G.

(Question) Center of a group is characteristic

(Solution) By Remark (above)

it is sufficies, to show $\sigma(Z(G)) \subset Z(G)$ $\forall$ $\sigma \in Aut(G)$, where G is group

Let $x \in \sigma(Z(G))$

$\Rightarrow$ $\exists$ $y \in Z(G)$ s.t $x = \sigma(y)$

To show: $x \in Z(G)$, consider an arbitrary $g \in G$

consider, $xg = \sigma(y)g$ $\quad$ $\left(\begin{array}{l}\because g \in G, \\ \sigma \in Aut(G) \\ \Rightarrow g = \sigma(g'), g' \in G\end{array}\right)$

$= \sigma(y)\sigma(g')$

$= \sigma(yg')$ ($\because \sigma$ is Homo.)

$= \sigma(g'y) \quad (\because y \in Z(G))$

$= \sigma(g')\,\sigma(y) \quad (\because \sigma \text{ is Homo.})$

$= g\,x$

Since, it is true for $\forall\, g \in G$

$\therefore x \in Z(G)$

$\Rightarrow \sigma(Z(G)) \subset Z(G)$

$\therefore Z(G)$ is characteristic subgrp.

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$\therefore \ |\phi(x)| \ldots$

(Question 2) If H is a unique subgrp of a group G of given order then H is characteristic

(Sol^n) Let H be subgrp (unique) of G order m

Let $\phi \in Aut(G)$

As $H \leq G$, ϕ is Automorphism

$\Rightarrow \phi(H) \leq G$

Since ϕ is bijective $\Rightarrow |\phi(H)| = |H| = m$

Also H is unique subgrp & $|\phi(H)| = |H|$

$\Rightarrow \phi(H) = H$

$\therefore$ H is characteristic of G.

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(1)

(Question) Prove that every subgrp of cyclic grp is characteristic.

(Sol^n) (a) Let G be finite cyclic grp of order 'n'
then $G \simeq Z_n$ & $H \leq G$
Since Z_n is cyclic grp of order 'n', then there exists unique subgrp of each allowed order
i.e if $m|n$ then $\exists$ unique subgrp of order m in Z_n

By above result
H is characteristic subgrp of G.

(b) Let G be cyclic grp (infinite)
then $Aut(G) = \{ I, T_{a^{-1}} ; a \mapsto a^{-1} \}$ (already proved)
let $H \leq G$
& $I(H) = H$ $(I \in Aut(G))$
let $a \in H \Rightarrow a^{-1} \in H$ $(\because H \leq G)$
$\Rightarrow T_{a^{-1}}(H) = H$
$\therefore$ $I(H) = H$ & $T_{a^{-1}}(H) = H$ for all $Aut(G)$
$\therefore$ H is characteristic subgrp
Hence, $\therefore$ Every subgrp of cyclic grp is characteristic subgrp.

(Question) Prove that characteristic property is transitive
i.e If ~~H~~ is characteristic subgrp of K & K is characteristic of G then ~~H~~ is characteristic subgrp of G.
H is characteristic subgrp of G.

(Solution) to show! H is characteristic subgrp of G
i.e $\phi(H) \subset H$ $\forall$ $\phi \in Aut(G)$
let $\phi \in Aut(G)$ be arbitrary
As K is char. subgrp of $G \Rightarrow \phi(K) = K$ $(\because \phi \in Aut(G))$
$\Rightarrow \phi_K : K \to K$ (restrict ϕ to K)
s.t $\phi_K(x) = x$ $\forall$ $x \in K$

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then ϕ_K is also Automorphism of K

[since ϕ is 1-1, onto, $\phi(K) = K \Rightarrow \phi_K(K) = K$
& ϕ_K is restriction to K so, 1-1]

$\therefore \phi_K \in Aut(K)$

$\therefore \phi_K \in$ ~~⊕~~ K & $\phi_K \in Aut(K)$

As, H is char subgrp of K

$\Rightarrow \phi_K(H) = H$

$\Rightarrow \phi(H) = H$ $\quad (\because \phi_K(x) = \phi(x) \ \forall x \in K)$

$\Rightarrow H$ is char. subgrp of G.

(Question) If H is char. subgrp of G then $H \triangleleft G$.

(Solution) let $h \in H$ & $g \in G$ be arbitrary elements

As $g \in G$

define $T_g : G \to G$

as $T_g(x) = gxg^{-1} \ \forall x \in G$

clearly $T_g \in Aut(G)$

Now $T_g(h) = ghg^{-1}$ —①

As H is char. subgrp of G

$\Rightarrow T_g(H) \subset H$ —②

from ① & ②

$T_g(h) = ghg^{-1} \in H$

$\Rightarrow ghg^{-1} \in H$

$\therefore H \triangleleft G$

(For normal subgrp $ghg^{-1} \in H \ \forall g \in G, h \in H$)

Converse Need Not be true

let $G = \{e, a, b, ab\}$ Kelvin group

s.t $a^2 = b^2 = e$, $ab = ba$

let $H = \{e, a\}$

then $H < G$

$i_G(H) = \frac{|G|}{|H|} = \frac{4}{2} = 2 \Rightarrow H \triangleleft G$

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Define $T : G \to G$
as $e \mapsto e$, $a \mapsto b$, $b \mapsto a$, $ab \mapsto ab$
clearly $T \in Aut(G)$
and $T(H) = \{e, b\}$ (as $H = \{e, a\}$)
and $T(H) \neq H$
$\therefore$ H is not a chara. subgrp of G.

Commutator subgrp

The commutator subgrp G' of a group G is the subgrp generated by the set
$\{x^{-1}y^{-1}xy \mid x, y \in G\}$ (the element $x^{-1}y^{-1}xy$ is called commutator)
In other words, every element of G' has form
$a_1^{i_1} a_2^{i_2} \cdots a_k^{i_k}$, where each a_j has form
$x^{-1}y^{-1}xy$, each $i_j = \pm 1$, k is any positive integer.

(1) Prove that G' is char. subgrp of G.
(Sol^n) to show: $\phi(G') \subset G'$ $\forall$ $\phi \in Aut(G)$
Let $\phi \in Aut(G)$ be arbitrary
Let $c = x^{-1}y^{-1}xy$ for some $x, y \in G$ ~~be element of G~~
$\Rightarrow \phi(c) = \phi(x^{-1}y^{-1}xy)$
$= \phi(x^{-1})\phi(y^{-1})\phi(x)\phi(y)$ ($\because \phi \in Aut(G)$)
$= (\phi(x))^{-1}(\phi(y))^{-1}\phi(x)\phi(y)$ ($\because \phi \in Aut(G)$)
$= x_1^{-1}y_1^{-1}x_1y_1$ (where $\phi(x) = x_1$, $\phi(y) = y_1$, where $x, y \in G$)
$\Rightarrow \phi(c) = x_1^{-1}y_1^{-1}x_1y_1$
$\Rightarrow \phi(c)$ ~~is~~ is commutator

Now, $c^{-1} = (x^{-1}y^{-1}xy)^{-1}$
$= (xy)^{-1}(x^{-1}y^{-1})^{-1}$
$= y^{-1}x^{-1}yx$

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$\because c^{-1}$ is commutator of G

let $\alpha \in G'$

$\Rightarrow \alpha = c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k}$ where c_i has form $x^{-1}y^{-1}xy$, $i_j = \pm 1$ & $k \in \mathbb{Z}^+$

$\Rightarrow \phi(\alpha) = \phi(c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k})$

$= \underbrace{\phi(c_1^{i_1})\phi(c_2^{i_2}) \cdots \phi(c_k^{i_k})}$

$=$ commutator (as above proved)

$\Rightarrow \phi(G') \subset G' \quad \forall \phi \in Aut(G)$

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