Linear Transformations:

Let V and W be vector spaces over a same field F. A function $T: V \to W$ is called a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

a) $T(x+y) = T(x) + T(y)$

b) $T(cx) = cT(x)$

Examples:

① Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ as

$$T(a_1, a_2) = (2a_1 + a_2, a_1)$$

To show that T is linear.

Solⁿ Let $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$, where

$$x = (a_1, a_2), \quad y = (b_1, b_2)$$

$\therefore T(x) = (2a_1 + a_2, a_1), \quad T(y) = (2b_1 + b_2, b_1)$

i) $T(x+y) = T((a_1, a_2) + (b_1, b_2))$

$$= T(a_1 + b_1, a_2 + b_2)$$

$$= (2(a_1 + b_1) + a_2 + b_2, a_1 + b_1)$$

[by def. of T]

$$= (2a_1 + a_2, a_1) + (2b_1 + b_2, b_1)$$

$$= T(a_1, a_2) + T(b_1, b_2)$$

$$= T(x) + T(y)$$

ii) $T(cx) = T(c(a_1,a_2)) = T(ca_1, ca_2)$

$$= (2(ca_1)+ca_2,\ ca_2)$$

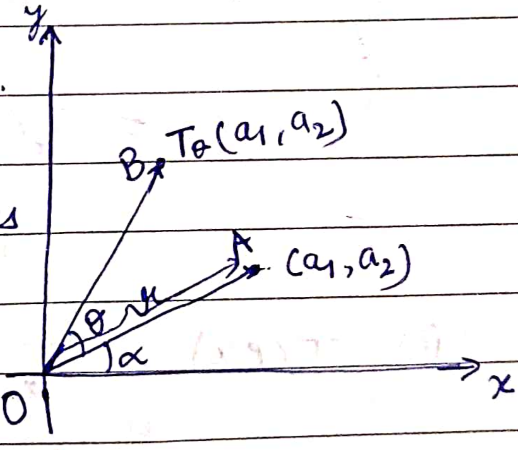
$$= c(2a_1+a_2,\ a_2)$$

$$= cT(a_1,a_2) = cT(x)$$

Hence T is a linear transformation.

Ex ② For any θ, define $T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $T_\theta(a_1,a_2)$ is the vector obtained by rotating (a_1,a_2) anticlockwise by θ if $(a_1,a_2) \neq (0,0)$ and $T_\theta(0,0) = (0,0)$. Then $T_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation that is called the rotation by θ.

Let $(a_1,a_2) \in \mathbb{R}^2$ be fixed. And let α be the angle that (a_1,a_2) makes with the +ve x-axis. And $OA = r = \sqrt{a_1^2+a_2^2}$.



Then

$$a_1 = r\cos\alpha, \quad a_2 = r\sin\alpha$$

Also, $T_\theta(a_1,a_2)$ has length r and makes an angle $\alpha+\theta$ with +ve x-axis. Thus

$$T_\theta(a_1,a_2) = (r\cos(\alpha+\theta),\ r\sin(\alpha+\theta))$$

$$= (r\cos\alpha\cdot\cos\theta - r\sin\alpha\cdot\sin\theta,\ r\cos\alpha\sin\theta + r\sin\alpha\cos\theta)$$

$$= (a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta)$$

Now will show that T_θ is linear.

Let $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$, where
$x = (a_1, a_2)$ and $y = (b_1, b_2)$

$\therefore T(x) = T(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$

$T_\theta(y) = T_\theta(b_1, b_2) = (b_1\cos\theta - b_2\sin\theta, b_1\sin\theta + b_2\cos\theta)$

(i) $T_\theta(x+y) = T_\theta((a_1, a_2) + (b_1, b_2))$

$= T_\theta(a_1 + b_1, a_2 + b_2)$

$= ((a_1+b_1)\cos\theta - (a_2+b_2)\sin\theta, (a_1+b_1)\sin\theta + (a_2+b_2)\cos\theta)$

$= (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$
$+ (b_1\cos\theta - b_2\sin\theta, b_1\sin\theta + b_2\cos\theta)$

$= T_\theta(a_1, a_2) + T_\theta(b_1, b_2)$

$= T(x) + T(y)$

(ii) $T_\theta(cx) = T_\theta(c(a_1, a_2)) = T_\theta(ca_1, ca_2)$

$= (ca_1\cos\theta - ca_2\sin\theta, ca_1\sin\theta + ca_2\cos\theta)$

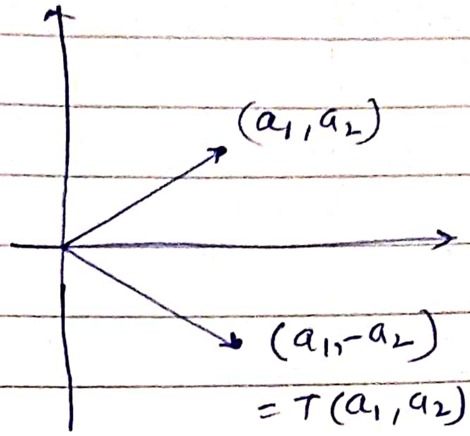
$= c(a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$

$= c\,T_\theta(a_1, a_2)$

$= c\,T_\theta(x)$

Ex 3 Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$.
T is called reflection about the x-axis

Now we will show that
T is linear.
Let $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$,
where $x = (a_1, a_2)$, $y = (b_1, b_2)$



$\therefore T(a_1, a_2) = (a_1, -a_2)$
$T(b_1, b_2) = (b_1, -b_2)$

(i) Now $T(x+y) = T((a_1, a_2) + (b_1, b_2))$
$= T(a_1 + b_1, a_2 + b_2)$
$= (a_1 + b_1, -(a_2 + b_2))$ [by def. of T]
$= (a_1, -a_2) + (b_1, -b_2)$
$= T(a_1, a_2) + T(b_1, b_2)$
$= T(x) + T(y)$

Ex 4 Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$.
T is called the projection on the x-axis.

TST: T is linear