Normal Modes

We shall now obtain the normal modes of transverse vibrations of a uniform string fixed at a both ends and stretched with a tension.

Consider a uniform string of length L and linear density μ stretched along the x-axis with a tension T and fixed at its ends $x = 0$ and $x = L$. Equation (6.6) describes the motion of any part of the string lying between $x = 0$ and $x = L$. In order to find the normal modes, we assume the existence of a normal mode at angular frequency ω and phase constant ϕ. This means that every particle of the string executes SHM of angular frequency ω and phase constant ϕ. Thus, for a normal mode we have

$$\boxed{y(x, t) = A(x) \cos (\omega t + \phi)} \tag{6.8}$$

These are infinite number of equations one for each particle characterized by its x value in the range 0 and L. The variable $y(x, t)$ is the displacement at time t of a particle located at x and $A(x)$ is the amplitude of its motion. The amplitudes of all particles of the string will determine the shape or configuration of the mode. Differentiating twice with respect to x and t we have

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{d^2 A(x)}{dx^2} \cos (\omega t + \phi)$$

and

$$\frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A(x) \cos (\omega t + \phi)$$

Notice that, since $A(x)$ is by definition a function x only, we can write the total derivative $\dfrac{d^2 A(x)}{dx^2}$ instead of a partial derivative. Substituting these derivatives in Eq. (6.6) gives

$$-\omega^2 A(x) \cos (\omega t + \phi) = v^2 \frac{d^2 A(x)}{dx^2} \cos (\omega t + \phi)$$

or

$$\boxed{\frac{d^2 A(x)}{dx^2} = -\frac{\omega^2}{v^2} A(x) = -k^2 A(x)} \tag{6.9}$$

where $\boxed{k = \omega/v.}$ The parameter k will be identified to be the *wave number* of the wave (see Chap. 7). At the moment k just stands for ω/v where

$$v = \sqrt{\frac{T}{\mu}}.$$

Equation (6.9) governs the shape of the mode. This is the familiar [differ]ential equation of SHM except that it represents oscillation in space (x) rather than in time (t). The general form of the harmonic oscillation in space can be written as

$$A(x) = A \sin kx + B \cos kx \tag{6.10}$$

where A and B are undetermined constants.

The general solution for the displacement $y(x, t)$ of the string, in a given mode, is obtained by using Eq. (6.10) in Eq. (6.8).

$$y(x, t) = (A \sin kx + B \cos kx) \cos (\omega t + \phi) \tag{6.11}$$

Boundary Conditions. Equation (6.11) is a bit too general because the boundary conditions have not been used so far. Our string is fixed at both ends. Suppose the string has total length L and the ends of the string are at $x = 0$ and $x = L$. Since these ends are rigidly fixed, there can be no displacement at these ends. In other words, the boundary conditions are

$$y(0, t) = y(L, t) = 0 \text{ for all values of } t$$

Using the first boundary condition, namely $y(0, t) = 0$ for all t in Eq. (6.11) we have

$$B = 0$$

Thus for a string fixed at $x = 0$, Eq. (6.11) reduces to

$$y(x, t) = A \sin kx \cos (\omega t + \phi) \tag{6.12}$$

Normal Mode Frequencies. The frequencies of the normal modes of transverse vibrations of the string can be obtained by using the second boundary condition, namely $y(L, t) = 0$ for all t, in Eq. (6.12) This requires

$$A \sin kL = 0$$

This equation is satisfied by choosing $A = 0$. But this corresponds to a trivial situation of a string permanently at rest. Hence the only way we can satisfy the boundary condition at $x = L$ is to have

$$\sin kL = 0$$

or

$$kL = n\pi$$

where n is an integer having values 1, 2, 3, ..., ∞. Thus

$$\boxed{k = \frac{n\pi}{L}} \tag{6.13}$$

We have excluded the case $n = 0$, i.e. $k = 0$, because this case also corresponds to an uninteresting situation of a string permanently at rest as is obvious from Eq. (6.12). Notice that the condition that the string is fixed at $x = L$ permits only some values of k, namely, those given by Eq. (6.13). But $k = \omega/v$, where $v = \sqrt{\dfrac{T}{\mu}}$ and ω is the angular frequency of the normal mode. The fact that only definite values of k [dictated by Eq. (6.13)] are permitted implies that only definite values of ω are allowed. These values are given by

$$\omega = kv$$

or

$$\boxed{\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}} \tag{6.14}$$

with $n = 1, 2, 3, \ldots, \infty$. Here we have used subscript n to indicate the value of ω for a particular integral value of n. Equation (6.14) gives the angular frequencies of the normal modes for transverse vibrations of a string fixed at both ends. The corresponding frequencies (in hertz) of the modes are given by

$$\nu_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \tag{6.15}$$

The mode with $n = 1$ is called the *fundamental* mode; its frequency is

$$\boxed{\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}}$$

The modes with $n = 2, 3, 4, \ldots$ are *harmonics* of the fundamental frequencies ν_1 which follows from the fact that

$$\nu_2 = 2\nu_1,\ \nu_3 = 3\nu_1,\ \nu_4 = 4\nu_1, \text{ etc.}$$

Thus the string has an infinite number of possible frequencies of vibration which are harmonics of the fundamental frequency ν_1. The fact that the mode frequencies ν_2, ν_3, etc. are harmonics of the fundamental mode frequency ν_1 is a result of our assumption that the string is perfectly uniform and flexible. Real strings do not strictly obey this simple sequence of frequencies The reason is that the strings in real physical systems (such as the strings of a violin or a piano) are not perfectly uniform and flexible.

Normal Mode Shapes. Equation (6.12) gives the displacement of the particles of the string in a normal mode. For the nth mode this equation reads

$$y_n(x, t) = A_n \sin k_n x \cos(\omega_n t + \phi_n) \tag{6.16}$$

where $k_n = \dfrac{n\pi}{L}$ and $\omega_n = k_n v = k_n\sqrt{\dfrac{T}{\mu}}$. The constants A_n and ϕ_n are to be determined from the initial conditions. We shall subsequently consider a set of initial conditions. Let us now obtain the shape of the first few modes.

In the fundamental mode ($n = 1$), also called the first harmonic, the particle displacements are given by

$$y_1(x, t) = A_1 \sin k_1 x \cos(\omega_1 t + \phi_1)$$

or

$$y_1(x, t) = A_1 \sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t + \phi_1)$$

This equation gives the displacements (in the fundamental mode) of all particles of the string (i e. all x values) as a function of time t. Notice that $y = 0$ at $x = 0, L$. There is no other value of x in the range 0 to L where y_1 can vanish. The displacement y_1 is maximum at $x = L/2$ for a given value of t. Figure 6.2 (a) shows a plot of $y_1(x, t)$ against x for different values of t. This is the shape of the fundamental mode. When the string is vibrating in one segment, as shown in Fig. 6.2 (a) its frequency of vibration is ν_1 given by

$$\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

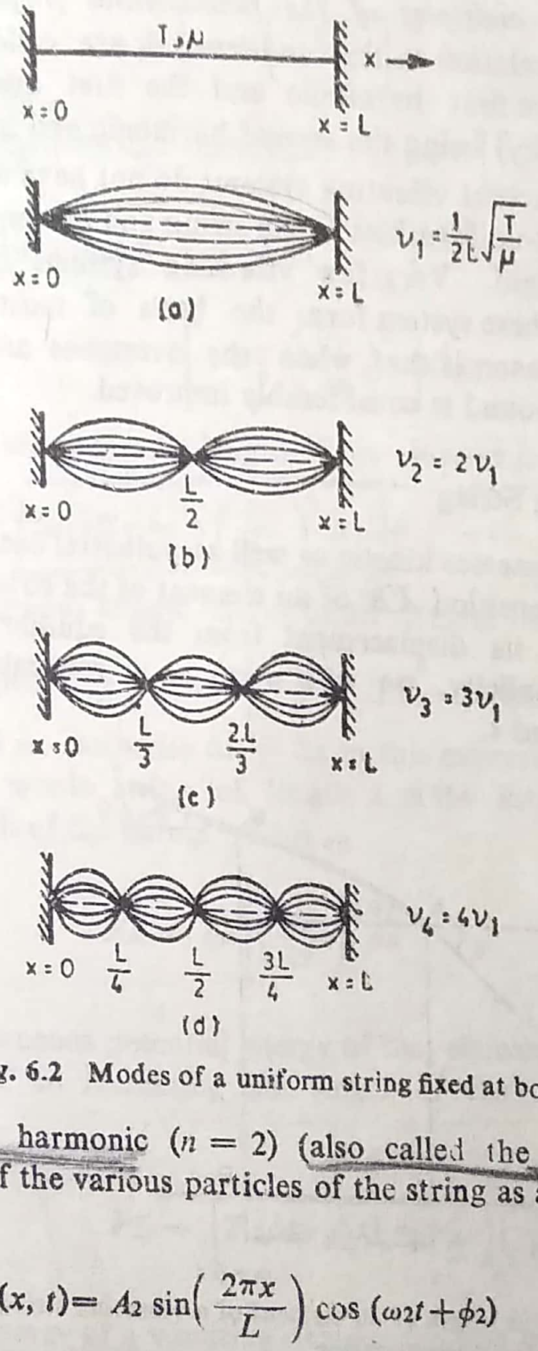

Fig. 6.2 Modes of a uniform string fixed at both ends

In the second harmonic ($n = 2$) (also called the first overtone), the displacements of the various particles of the string as a function of time are given by

$$y_2(x, t) = A_2 \sin\left(\frac{2\pi x}{L}\right) \cos(\omega_2 t + \phi_2)$$

Notice that y_2 is zero at $x = 0$, $L/2$ and L. Figure 6.2 (b) gives the shape of this mode; the string now vibrates in two segments at a frequency ν_2 which is twice the frequency ν_1 of the fundamental mode. Figures 6.2 (c) and (d) show the next two harmonics. These figures reveal that there are certain points on the string which are permanently at rest; the number of these points depends upon the number of the mode under study. These points are called *nodes*. The points where the displacement is maximum (at a given time) are called *antinodes*. In the next chapter we shall show that the *standing waves* on a string are nothing but its normal modes.

It may be remarked that Eq. (6.15) for allowed frequencies expresses a very important property of a uniform flexible string stretched between rigid supports. It states that the frequencies of all the overtones of such a string are *integral multiples of the fundamental frequency*. Overtones bearing this simple relation to the fundamental are called *harmonics*; the fundamental being the first harmonic and the first overtone (twice the fundamental frequency) being the second harmonic and so on.

As stated earlier, actual vibrating systems do not have exactly harmonic overtones due to non-uniformities in the string and the supports at its ends being not perfectly rigid. Very few vibrating systems have nearly harmonic overtones. These system form the basis of most of the musical instruments. The reason is that, when the overtones are harmonic, the tonal quality of the sound is considerably improved.