Corollary:

For every prime p, $\mathbb{Z}_p$, the ring of integers modulo p, is a field.

Proof: $\mathbb{Z}_p = \{0, 1, 2, \dots, p-1\}$

TS $\mathbb{Z}_p$ is field.

$\because$ $\mathbb{Z}_p$ is finite, So only need to show that $\mathbb{Z}_p$ is an I.D.

Clearly, $\mathbb{Z}_p$ is a commutative ring with unity $(\because 1 \in \mathbb{Z}_p)$

It is enough to show that $\mathbb{Z}_p$ has no zero divisors.

Suppose that $a, b \in \mathbb{Z}_p$ and $ab = 0$

TS $a = 0$ or $b = 0$.

Now $a, b \in \mathbb{Z}_p$ and $ab = 0$

$\Rightarrow ab \equiv 0 \pmod{p}$

$\Rightarrow ab = pk$ for some $k \in \mathbb{N}$

$\Rightarrow p \mid ab$

$\Rightarrow p \mid a$ or $p \mid b$ (using euclid's lemma)

$\Rightarrow a \pmod p = 0$ or $b \pmod p = 0$

$\Rightarrow a = 0$ or $b = 0$ in $\mathbb{Z}_p$.

$\Rightarrow \mathbb{Z}_p$ has no zero divisors

$\Rightarrow \mathbb{Z}_p$ is an I.D.

$\Rightarrow \mathbb{Z}_p$ is a field ($\because \mathbb{Z}_p$ is finite).

Characteristic of a Ring:-

The characteristic of a ring R is the least positive integer n such that

$$nx = 0 \quad \forall x \in R.$$

If no such integer exists, we say that R has characteristic 0.

Notation $\Rightarrow$ Char. R

Result :

① Let R be a ring with unity 1.

If 1 has infinite order under addition

then char. $R = 0$.

If 1 has finite order n under addition,

then char. $R = n$

② Characteristic of an Integral Domain is 0 or prime.

③ Characteristic of a field is 0 or prime.

Example : ① $R = (\mathbb{R}, +, \cdot)$

Non. $nx = 0$

char. $R = 0$

② $R = (\mathbb{Z}_6, +_6, \cdot_6)$, $|1| = 6$ (under addition)

$\text{char}. R = 6.$

③ $R = (\mathbb{Z}_5, +_5, \cdot_5)$ $\quad$ $\text{char}. R = 5$ (Prime).