6th October 2020

Maximum - minimum

Examples:

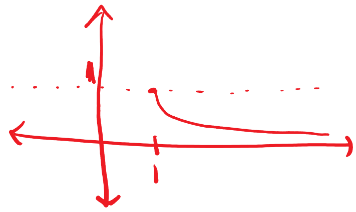
$f : A \to \mathbb{R}$

$u \in A$, $f(u) \geq f(x)$ $\forall x \in A$ f has a abs. max. on A.

'u' point of abs max.

$f(x) = \frac{1}{x}$ on $(0, \infty)$

Example ① $f(x) = \frac{1}{x}$ on $\underbrace{[1, \infty)}_{A}$



$f(u) = 1 \geq f(x)$

$x \geq 1$, $\frac{1}{x} \leq 1$

$u = 1$

$f(1) \geq f(x)$ $\forall x \in A$

'1' is point of abs max.

& $f(1)$ i.e 1 is the abs. max of f on $[1, \infty)$

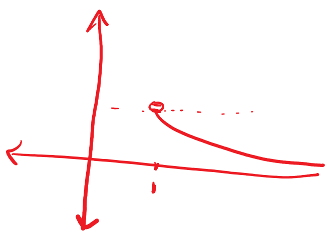
$f(x) = \frac{1}{x}$ on $[2, \infty)$ Abs max? $\frac{1}{2}$

pt of abs max? '2'

② $f(x) = \frac{1}{x}$ on $(1, \infty) = A$

Does f posses abs. max on A?

No



$f(x) \neq 1$ for any $x \in (1, \infty)$

$\sup \{ f(x) \mid x \in A \} = 1$

$\sup \{ f(x) \mid x \in (1, \infty) \}$

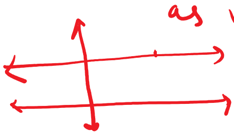
$\sup \{ \frac{1}{x} \mid x \in (1, \infty) \}$

$= 1$

$\frac{1}{x} = 1$

$\therefore$ f does not possess abs max on A

③ Give an example of a f^n on $\mathbb{R}$ for which all points of $\mathbb{R}$ are the points of abs max. as well abs min.



$\to$ Constant f^n

as $\forall x \in D$, $f(x) = k$ ($k \in \mathbb{R}$ is any constant)

$f(x)$ is max as well min for at $x \in D$

Theorem 5.3.4 Maximum-minimum Theorem

Theorem 5.3.: Maximum-minimum Theorem

$\boxed{\text{cts fn on closed \& bdd interval } I \text{ possesses abs max/min on } I}$

Let $I = [a, b]$ be closed & bdd & $f : I \to \mathbb{R}$ be cts on I.
Then f has an absolute max & an abs min. on I

Proof: Let $f : I \xrightarrow{cts} \mathbb{R}$, $I = [a, b]$

(we'll show f has abs max on I i.e $\exists\, u \in I : f(u) \geq f(x)\ \forall x \in I$)

Let $f(I) = \{ f(x) \mid x \in I \}$

As f is cts & I is closed & bdd $\therefore$ by boundedness thm f is bdd on I

$\Rightarrow f(I)$ is a bdd set $(f(I) \neq \phi)$

$\therefore f(I)$ is a non-empty bdd above set

$\Rightarrow \sup f(I)$ exists (by order completeness property)

$S = f(I)$

Let $M = \sup f(I)$

$\Rightarrow f(x) \leq M \quad \forall x \in I$ — ①

Target $f(x^*)$, $x^* \in I$

$\forall n \in \mathbb{N}$, $M - \frac{1}{n}$ is not an up bd of $f(I)$ (S)

$\Rightarrow \exists\, x_n \in I : \quad f(x_n) > M - \frac{1}{n}$

$S = \{ f(x) \mid x \in I \}$

i.e $\forall n$, $\exists\, x_n \in I$:

$$M - \frac{1}{n} < f(x_n) \leq M \quad \text{—— ②}$$

Now (x_n) is a seq in I

$\Rightarrow (x_n)$ is bdd seq $\quad (\because a \leq x_n \leq b)$

By Bolzano Wierstrass (x_n) possesses a convergent subseq, say (x_{n_k})

Let $(x_{n_k}) \to x^*$

$\forall k$
$(a \leq x_{n_k} \leq b$
$a \leq \lim x_{n_k} \leq b)$

As $(x_{n_k}) \in I \Rightarrow x^* \in I$

$x^* \in I$ & f is cts on I

by seq criterion, $f(x_{n_k}) \to f(x^*)$

$S \neq \phi$
$M = \sup S$
① $s \leq M \quad \forall s \in S$ (up. bd)
② M is least bound of S
If M' is another bound then $M \leq M'$
or
$\varepsilon > 0$, $M - \varepsilon$ is NOT an up bd of S ($M - \varepsilon < M$)
i.e $\exists\, s \in S$, $s > M - \varepsilon$

$\{1, 2, 3\}$ $3 > 2.5$ NOT an up. bd.

$S_1 = \{1, 2, 3, 4, \ldots\}$
$\sup S_1$ does not exist

$S_2 = \{1, 2, 3, 4, \ldots, 10\}$
$\sup S_2 = 10$

$S_3 = [0, 100]$ or $[0, 100)$
$\sup S_3 = 100$

$x^* \in I$ & f is cts on I

by seq criterion, $f(x_{n_k}) \to f(x^*)$ —— ③

from ②, $\left(M - \frac{1}{n_k}\right) \leq f(x_{n_k}) \leq M \quad \forall k$

$\Rightarrow \quad M \leq \lim_{k \to \infty} f(x_{n_k}) \leq M$

$\Rightarrow \lim_{k \to \infty} f(x_{n_k}) = M$ —— ④

By uniqueness of limit of seq, using ③ & ④ we get

$f(x^*) = M$, where $x^* \in I$

i.e $\exists x^* \in I$: $f(x^*) = \sup f(I) \geq f(x) \quad \forall x \in I$

$\Rightarrow$ x^* is a pt of abs max.

& $f(x^*)$ is the abs max of f on I

$S_3 = [0, 100]$ or $[0, 100)$

$\sup S_3 = 100$

$S_4 = (0, \infty)$ ~~sup~~ $\sup S_4$ d.n.e

$M - \frac{1}{n} \leq f(x_n) \leq M \quad \forall n$

$a \leq b \leq a$

$b = a$

$f(I)$ bdd set $\to \sup f(I) = M$

$\to M - \frac{1}{n} < f(x_n) \leq M \longrightarrow f(x_n) \to M$

$x_n \in I \quad$ BW $\quad (x_{n_k}) \to x^* \in I$

$f(x_{n_k}) \longrightarrow f(x^*)$