Topology of a Metric Space

Open and Closed Sets

Definition1.

Let (X, d) be a metric space. The set

$$S(x_0, r) = \{x \in X : d(x_0, x) < r\}, \qquad \text{where } r > 0 \text{ and } x_0 \in X,$$

is called the **open ball** of radius r and centre x_0. The set

$$\bar{S}(x_0, r) = \{x \in X : d(x_0, x) \leq r\}, \qquad \text{where } r > 0 \text{ and } x_0 \in X,$$

is called the **closed ball** of radius r and centre x_0.

Example1.

The open ball $S(x_0, r)$ on the real line is the bounded open interval $(x_0 - r, \ x_0 + r)$ with midpoint x_0 and total length $2r$. Conversely, it is clear that any bounded open interval on the real line is an open ball. So the open balls on the real line are precisely the bounded open intervals. The closed balls $\bar{S}(x_0, r)$ on the real line are precisely the bounded closed intervals but containing more than one point.

Definition2.

Let (X, d) be a metric space. A **neighbourhood** of the point $x_0 \in X$ is any open ball in (X, d) with centre x_0.

Definition3.

A subset G of a metric space (X, d) is said to be **open** if given any point $x \in G$, there exists $r > 0$ such that $S(x, r) \subseteq G$, i.e., each point of G is the centre of some open ball contained in G. Equivalently, every point of the set has a neighbourhood contained in the set.

Theorem1.

In any metric space (X, d), each open ball is an open set.

Proof. First observe that $S(x,r)$ is nonempty, since $x \in S(x,r)$. Let $y \in S(x,r)$, so that $d(y,x) < r$, and let $r' = r - d(y,x) > 0$. We shall show that $S(y,r') \subseteq S(x,r)$, as illustrated in Fig. 2.5. Consider any $z \in S(y,r')$. Then we have

$$d(z,x) \leq d(z,y) + d(y,x) < r' + d(y,x) = r,$$

which means $z \in S(x,r)$. Thus, for each $y \in S(x,r)$, there is an open ball $S(y,r') \subseteq S(x,r)$. Therefore $S(x,r)$ is an open subset of X. □

////////////

Theorem2.

Let (X, d) be a metric space. Then

(i) $\emptyset$ and X are open sets in (X, d);
(ii) the union of any finite, countable or uncountable family of open sets is open;
(iii) the intersection of any finite family of open sets is open.

Proof. (i) As the empty set contains no points, the requirement that each point in $\varnothing$ is the centre of an open ball contained in it is automatically satisfied. The whole space X is open, since every open ball centred at any of its points is contained in X.

(ii) Let $\{G_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of open sets and $H = \cup_{\alpha \in \Lambda} G_\alpha$. If H is empty, then it is open by part (i). So assume H to be nonempty and consider any $x \in H$. Then $x \in G_\alpha$ for some $\alpha \in \Lambda$. Since G_α is open, there exists an $r > 0$ such that $S(x, r) \subseteq G_\alpha \subseteq H$. Thus, for each $x \in H$ there exists an $r > 0$ such that $S(x, r) \subseteq H$. Consequently, H is open.

(iii) Let $\{G_i : 1 \leq i \leq n\}$ be a finite family of open sets in X and let $G = \cap_{i=1}^{n} G_i$. If G is empty, then it is open by part (i). Suppose G is nonempty and let $x \in G$. Then $x \in G_j, j = 1, \ldots, n$. Since G_j is open, there exists $r_j > 0$ such that $S(x, r_j) \subseteq G_j, j = 1, \ldots, n$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then $r > 0$ and $S(x, r) \subseteq S(x, r_j), j = 1, \ldots, n$. Therefore the ball $S(x,r)$ centred at x satisfies

$$S(x, r) \subseteq \bigcap_{j=1}^{n} S(x, r_j) \subseteq G.$$

This completes the proof. □

Note.

The intersection of an infinite number of open sets need not be open.

To see why, let $S_n = S(0, \frac{1}{n}) \subseteq \mathbf{C}$, $n = 1, 2, \ldots$. Each S_n is an open ball in the complex plane and hence an open set in $\mathbf{C}$. However,

$$\bigcap_{n=1}^{\infty} S_n = \{0\},$$

which is not open, since there exists no open ball in the complex plane with centre 0 that is contained in $\{0\}$.

////////////

Theorem2.

A subset G in a metric space (X, d) is open if and only if it is the union of all open balls contained in G.

Proof. Suppose that G is open. If G is empty, then there are no open balls contained in it. Thus, the union of all open balls contained in G is a union of an empty class, which is empty and therefore equal to G. If G is nonempty, then since G is open, each of its points is the centre of an open ball contained entirely in G. So, G is the union of all open balls contained in it.

The converse follows immediately from **Theorem 1 and Theorem2.**

/////////

Example 1.

The open ball $S(x_0, r)$ in $\mathbf{R}^2$ with metric d_2 (see Example 1.2.2(iii)) is the inside of the circle with centre x_0 and radius r as in Fig. 2.1. Open balls of radius 1 and centre (0,0), when the metric is d_1 or d_∞ are illustrated in Figs. 2.2 and 2.3.

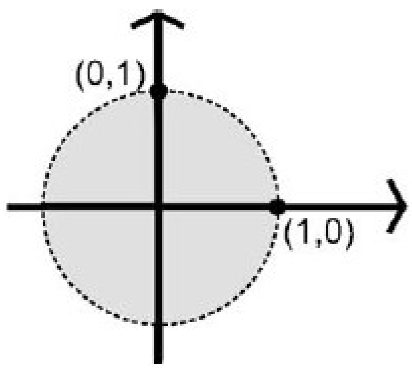

FIGURE 2.1

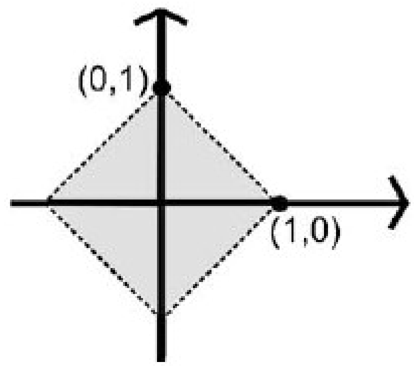

FIGURE 2.2

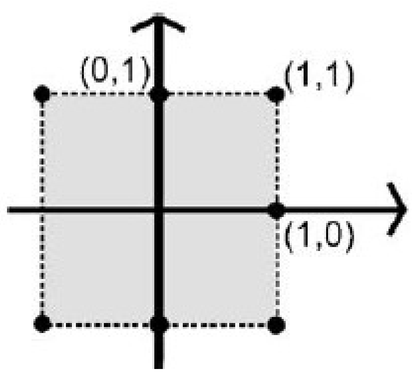

FIGURE 2.3

Example 2.

Consider the metric space $C_{\mathbf{R}}[a, b]$ of Example 1.2.2(ix). The open ball $S(x_0, r)$, where $x_0 \in C_{\mathbf{R}}[a, b]$ and $r > 0$, consists of all continuous functions $x \in C_{\mathbf{R}}[a, b]$ whose graphs lie within a band of vertical width $2r$ and is centred around the graph of x_0. (See Fig. 2.4.)

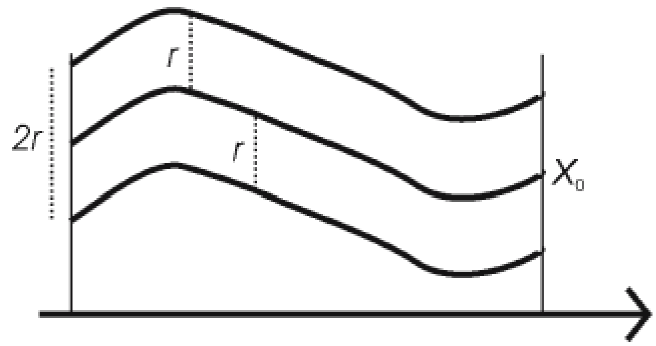

FIGURE 2.4

Definition4.

A subset G of a metric space (X, d) is said to be **open** if given any

point $x \in G$, there exists $r > 0$ such that $S(x, r) \subseteq G$, i.e., each point of G is the centre of some open ball contained in G. Equivalently, every point of the set has a neighbourhood contained in the set.

Theorem3.

In any metric space (X, d), each open ball is an open set.

Proof. First observe that $S(x,r)$ is nonempty, since $x \in S(x, r)$. Let $y \in S(x, r)$, so that $d(y, x) < r$, and let $r' = r - d(y, x) > 0$. We shall show that $S(y, r') \subseteq S(x, r)$, as illustrated in Fig. 2.5. Consider any $z \in S(y, r')$. Then we have

$$d(z, x) \leq d(z, y) + d(y, x) < r' + d(y, x) = r,$$

which means $z \in S(x, r)$. Thus, for each $y \in S(x, r)$, there is an open ball $S(y, r') \subseteq S(x, r)$. Therefore $S(x, r)$ is an open subset of X. □

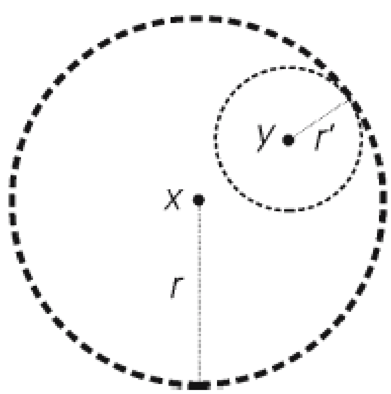

FIGURE 2.5

Example 3.

In a discrete metric space X, any subset G is open, because any $x \in G$ is the centre of the open ball $S(x, 1/2)$ which is nothing but $\{x\}$.

Example 4.

In ℓ_2, let $G = \{x = \{x_i\}_{i \geq 1}: \sum_1^\infty |x_i|^2 < 1\}$. Then G is an open subset of ℓ_2.

Indeed, $G = S(0, 1)$ is the open ball with centre $0 = (0, 0, \dots)$ and radius 1.

Theorem4. Let (X, d) be a metric space. Then

(i) $\varnothing$ and X are open sets in (X, d);
(ii) the union of any finite, countable or uncountable family of open sets is open;
(iii) the intersection of any finite family of open sets is open.

Proof. (i) As the empty set contains no points, the requirement that each point in $\varnothing$ is the centre of an open ball contained in it is automatically satisfied. The whole space X is open, since every open ball centred at any of its points is contained in X.

(ii) Let $\{G_\alpha: \alpha \in \Lambda\}$ be an arbitrary family of open sets and $H = \cup_{\alpha \in \Lambda} G_\alpha$. If H is empty, then it is open by part (i). So assume H to be nonempty and consider any $x \in H$. Then $x \in G_\alpha$ for some $\alpha \in \Lambda$. Since G_α is open, there exists an $r > 0$ such that $S(x, r) \subseteq G_\alpha \subseteq H$. Thus, for each $x \in H$ there exists an $r > 0$ such that $S(x, r) \subseteq H$. Consequently, H is open.

(iii) Let $\{G_i: 1 \leq i \leq n\}$ be a finite family of open sets in X and let $G = \cap_{i=1}^n G_i$. If G is empty, then it is open by part (i). Suppose G is nonempty and let $x \in G$. Then $x \in G_j, j = 1, \dots, n$. Since G_j is open, there exists $r_j > 0$ such that $S(x, r_j) \subseteq G_j, j = 1, \dots, n$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then $r > 0$ and $S(x, r) \subseteq S(x, r_j), j = 1, \dots, n$. Therefore the ball $S(x,r)$ centred at x satisfies

$$S(x, r) \subseteq \bigcap_{j=1}^{n} S(x, r_j) \subseteq G.$$

This completes the proof. □

Note.

The intersection of an infinite number of open sets need not be open.

To see why, let $S_n = S(0, \frac{1}{n}) \subseteq C$, $n = 1, 2, \ldots$. Each S_n is an open ball in the complex plane and hence an open set in C. However,

$$\bigcap_{n=1}^{\infty} S_n = \{0\},$$

which is not open, since there exists no open ball in the complex plane with centre 0 that is contained in $\{0\}$.

Theorem5. A subset G in a metric space (X, d) is open if and only if it is the union of all open balls contained in G.

Proof. Suppose that G is open. If G is empty, then there are no open balls contained in it. Thus, the union of all open balls contained in G is a union of an empty class, which is empty and therefore equal to G. If G is nonempty, then since G is open, each of its points is the centre of an open ball contained entirely in G. So, G is the union of all open balls contained in it. The converse follows immediately from

previous theorems.

//