UNIT 3 LINEAR PROGRAMMING – GRAPHICAL METHOD

Objectives

After studying this unit, you should be able to :

- Formulate management problem as a linear programming problem in suitable cases
- identify the characteristics of a linear programming problem
- make a graphical analysis of the problem
- solve the problem graphically
- identify various types of solutions
- explain various applications of linear programming in business and industry.

Structure

3.1 INTRODUCTION

Linear Programming is a versatile technique which can be applied to a variety of problems of management such as production, refinery operation, advertising, transportation, distribution and investment analysis. Over the years linear Programming has been found useful not only in business and industry but also in non-profit organisations such as government, hospitals, libraries and education. The technique is applicable in problems characterised by the presence of a number of decision variables, each of which can assume values within a certain range and affect their decision variables. The variables represent some physical or economic quantities which are of interest to the decision maker and whose domains are governed by a number of practical limitations or constraints. These may be due to availability of resources like men, material or money or may be due to a quality constraint or may arise from a variety of other reasons. The problem has a well defined objective. The common most objectives are maximisation of profit/contribution or minimisation of cost. Linear programming indicates the right combination of the various decision variables which can be best employed to achieve the objective taking full account of the practical limitations within which the problem must be solved.

The most important feature of linear programming is the presence of linearity in the problem. This will enable you to convert the objective to a linear function of the decision variables and the constraints into linear inequalities. The problem thus reduce to maximising or minimising a linear function subject to a number of linear inequalities. Although only graphical methods of solution are presented in this unit, very efficient computational procedures known as algorithms are available to solve linear programming problems. The emergence of computers has been helpful to solve these problems with a large number of decision variables and constraints.

Activity 1

Fill in the blanks

i) A linear programming problem has a well defined objective function which is .... and which is to be ....or ....

ii) The constraints in a linear programming problem arises due to limitation of These are linear ....or ....

iii) The solution of a linear programming problem indicates the right combination of ....which .... or .... the objective function satisfying the various .....: ....

3.2 FOMULATION OF A LINEAR PROGRAMMING PROBLEM

The formulation of a linear programming problem can be illustrated through what is known as a product mix problem. Typically, it occurs in a manufacturing industry where it is possible to manufacture a variety of products. Each of the products has a certain margin of profit per unit. These products use a common pool of resources - whose availability is limited. The linear programming technique identifies the combination of the products which will maximise the profit without violating the resource contraints. The formulation is illustrated with the help of following example.

Example 1

Suppose an organisation is manufacturing two products $P_1$and P_2. The profit per tonne of the two products are Rs. 50 and Rs. 60 respectively. Both the products require processing in three types of machine. The following Table indicates the available machine hours per week and the time required on each machine for one tonne of P_1 and P_2. Formulate this product mix problem in the linear programming form.

Table Showing the available machine capacities and machine hour requirement of the two products

Profit/tonne	Product 1 Rs. 50	Product 2 Rs. 60	Total available Machine hours/weeks
Machine 1		2	300
Machine 2		34	509
Machine 3		4	812

Solution

We introduce **decision variables** x_1 and x_2 indicating the amount of P_1 and P_2 to be included in the product mix. The **objective function** can be expressed as

$$50x_1 + 60x_2$$

which is to be maximised. Since one tonne of product P_1 requires two hours of processing in machine 1 while the corresponding requirement of P_2 is one hour the first **constraint** can be expressed as

$$2x_1 + x_2 \leq 300$$

Similarly, the constraints corresponding to machine 2 and machine 3 are

$$3x_1 + 4x_2 \leq 509$$

$$4x_1 + 7x_2 \leq 812$$

In addition there cannot be any negative production which may be stated algebraically as

$$x_1 \geq 0 \qquad x_2 \geq 0$$

Thus, the decision variables, the objective function and the constraints of the product mix problem have been identified. The problem can now be stated in the standard linear programming form as

Subject to :

$$2x_1 + x_2 \leq 300$$
$$3x_1 + 4x_2 \leq 509$$
$$4x_1 + 7x_2 \leq 812$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

This procedure is also commonly referred to as the formulation of the problem.

Activity 2

Himalayan Orchards have canned apple and bottled juice as its products Rs. 2 and Rs. 1 respectively per unit. The following Table indicates the margin labour, equipment and material to produce each product per unit.

	Bottled Juice	Canned Apple	Total
Labour (man hours)	3.0	2.0	12.0
Equipment (machine hours)	1.0	2.3	6.9
Material (unit)	1.0	1.4	4.9

Formulate the product mix which will Formulate the linear programming problem specifying maximise profit without exceeding the various levels of resources.

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Activity 3

A television manufacturer must decide how many black-and-white and how many colour sets he should produce for each day's sale so as to maximize his daily profit. Each day he has available the following supplies:

TV Chasis 24

Production hours 160

Colour Tubes 10

Each black-and-white set requires 5 production hours and yields a profit of Rs. 60. Each colour set requites 10 production hours and yields a profit of Rs. 150. Formulate the manufacturer's problem as a linear programming problem.

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3.3 FORMULATION WITH DIFFERENT TYPES OF CONSTRAINTS

The formulation of a linear programming problem when the constraints have emerged due to resource limitation has been illustrated in the previous section. These constraints are of "less than equal to" type. However, there may be constraints of other types as well. The formulation of linear programming problems with different types of constraints is illustrated in the following examples.

Example 2

Three nutrient components namely, thiamine, phosphorus and iron are found in a diet of two food items A and B. The amount of each nutrient (in milligrams per ounce i.e. mg/oz) is given below :

	A	B
Thiamine	0.15 mg/oz	0.10 mg/oz
Phosphorus	0.75 mg/oz	1.70 mg/oz
Iron	1.30 mg/oz	1.10 mg/oz

The cost of food A and B are Rs. 2 per oz. and Rs. 1.70 per oz. respectively. The minimum daily requirements of these nutrients are at least 1.00 mg. of thiamine, 7.50 mg of phosphorus and 10.00 mg of iron. Write the problem in the linear programming form.

Solution

Let us define by x_1 and x_2 the number of units (ounces) of A and B respectively purchased everyday. Since the purpose is to minimise the total cost of the food items and to satisfy the minimum daily requirement of nutrient the linear programming problem is given by

$$\text{Minimise } 2x_1 + 1.7x_2$$

Subject to :

$$0.15x_1 + 0.10x_2 \geq 1.0$$
$$0.75x_1 + 1.70x_2 \geq 7.5$$
$$1.30x_1 + 1.10x_2 \geq 10.0$$
$$x_1 \geq 0, \quad x_2 \geq 0.$$

Example 3

An oil refinery can blend three grades of crude oil to produce quality R and quality S petrol. Two possible blending processes are available. For each production run the older process uses 5 units of crude A, 7 units of crude B and 2 units of crude C to produce 9 units of R and 7 units of S. The newer process uses 3 units of crude A, 9 units of crude B and 4 units of crude C to produce 5 units of R and 9 units of S petrol.

Because of prior contract commitments the refinery must produce at least 500 units of R and at least 300-units of S for the next month. It has available 1500 units of crude A, 1900 units of crude B and 1000 units of crude C. For each unit of R the refinery receives Rs. 60 while for each unit of S it receives'Rs. 90. Find out the linear programming formulation of the problem so as to maximise the revenue.

Solution

The decision variables in this case are

$$x_1 = \text{no. of production runs of the older process.}$$
$$x_2 = \text{no. of production runs of the newer process.}$$

Fractional runs are permissible in blending processes. Then from the given conditions the problem can be formulated as

$$\text{Maximise } 60(9x_1 + 5x_2) + 90(7x_1 + 9x_2)$$
$$= 1170x_1 + 1110x_2$$

Subject to:

$$9x_1 + 5x_2 \geq 500 \text{ (commitment on R)}$$
$$7x_1 + 9x_2 \geq 300 \text{ (commitment on S)}$$
$$5x_1 + 3x_2 \leq 1500 \text{ (availability of A)}$$
$$7x_1 + 9x_2 \leq 1900 \text{ (availability of B)}$$
$$2x_1 + 4x_2 \leq 1000 \text{ (availability of C)}$$
$$x_1, x_2 \geq 0$$

In the example 2 of this section the linear programming problem has only "greater than or equato" type constraints. In the example 3 the problem has both "greater than or equal to" type and "less than or equal to" type constraints.

Activity 4

A company owns two flour mills, A and B, which have different production capacities, for high, medium and low grade flour. This company has entered a contract to supply flour to a firm every week with at least 12, 8 and 24 quintals of high, medium and. low grade respectively. It costs the company Rs. 1000 and Rs. 800 per day to run mill A and B respectively. On a day, mill A produces 6, 2 and 4 quintals of high, medium and low grade flour respectively. Mill B produces 2, 2 and 12 quintals of high, medium and low grade flour respectively. How many days per week should each mill be operated in order. to meet the contract order most economically.

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Activity 5

A scrap metal dealer has received a bulk order from a customer for a supply of at least 2000 kg of scrap metal. The customer has specified that at least 1000 kgs of the order must be of high quality copper that can be melted easily and can be used to produce tubes. Further, the customer has specified that the order should not contain more than 200 kg. of scrap which are unfit for commercial purposes. The scrap metal dealer purchases scrap from two different sources in an unlimited quantity with the following percentages (by weight) of high quality copper and unfit scrap.

	Source A	**Source B**
Copper	40%	75%
Unfit Scrap	7.5%	10%

The cost of metal purchased from Source A and Source B are Rs. 12.50 and Rs. 14.50 per kg. respectively. Determine the optimum quantities of metal to be purchased from the two Sources by the scrap metal dealer so as to minimise the total cost.

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3.4 GRAPHICAL ANALYSIS

Linear programming with two decision variables can be analysed graphically. The graphical analysis of a linear programming problem is illustrated with the help of the following example of product mix introduced in Section 3.2.

Maximise $50x_1 + 60x_2$

Subject to :

$$2x_1 + x_2 \leqslant 300$$
$$3x_1 + 4x_2 \leqslant 509$$
$$4x_1 + 7x_2 \leqslant 812$$
$$x_1 \geqslant 0, \ x_2 \geqslant 0$$

At first we draw the line $2x_1 + x_2 = 300$ which passes through the points (0, 300) and (150, 0).

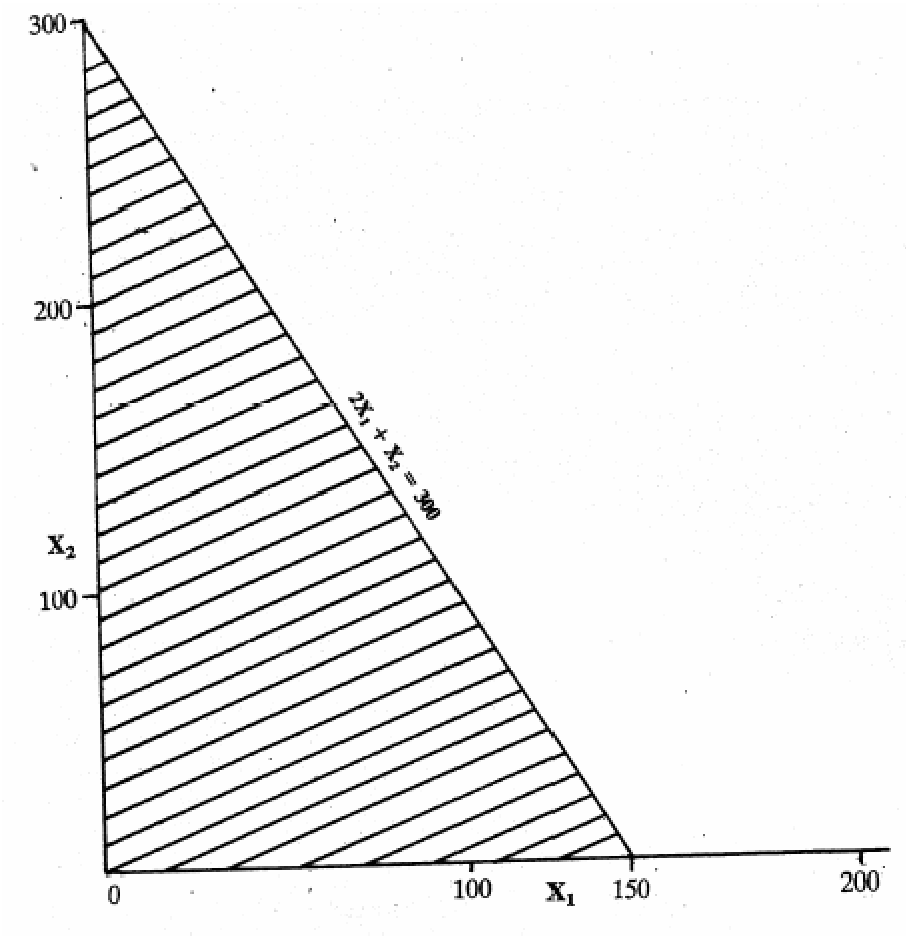

Figure 3.1 : Closed Half Plane

A linear inequality in two variables is known as a half plane. The corresponding equality or the line is known as the boundary of the half plane. The half plane along with its boundary is called a closed half plane. We must decide on which side of that line $2x_1 + x_2 = 300$ the half plane is located. An easy way is to solve the inequality for x2.

$$x_1 \leqslant 300 - 2x_2$$

For fixed x_1, the ordinates satisfying this inequality are smaller than the corresponding ordinate on the line and thus the inequality is satisfied for all points below the line. This is the shaded region as indicated in Figure 3.1.

Similarly, you may determine the closed half planes corresponding to the inequalities $3x_1 + 4x_2 \leqslant 509$ and $4x_1 + 7x_2 \leqslant 812$ (Figures 3.2 and 3.3).

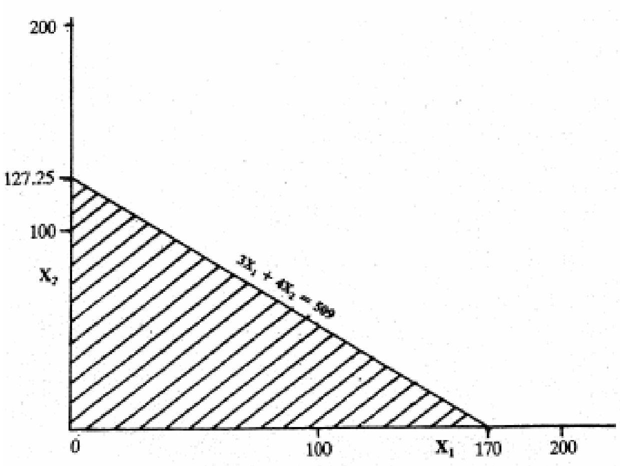

Figure 3.2 : Closed Half Plane

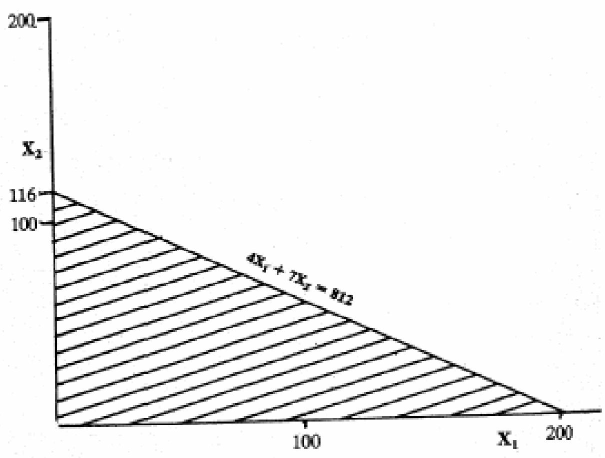

Figure 3.3 : Closed Half Plane

Since all the three constraints must be satisfied simultaneously we consider the intersection of these three closed half planes in Figure 3.4.

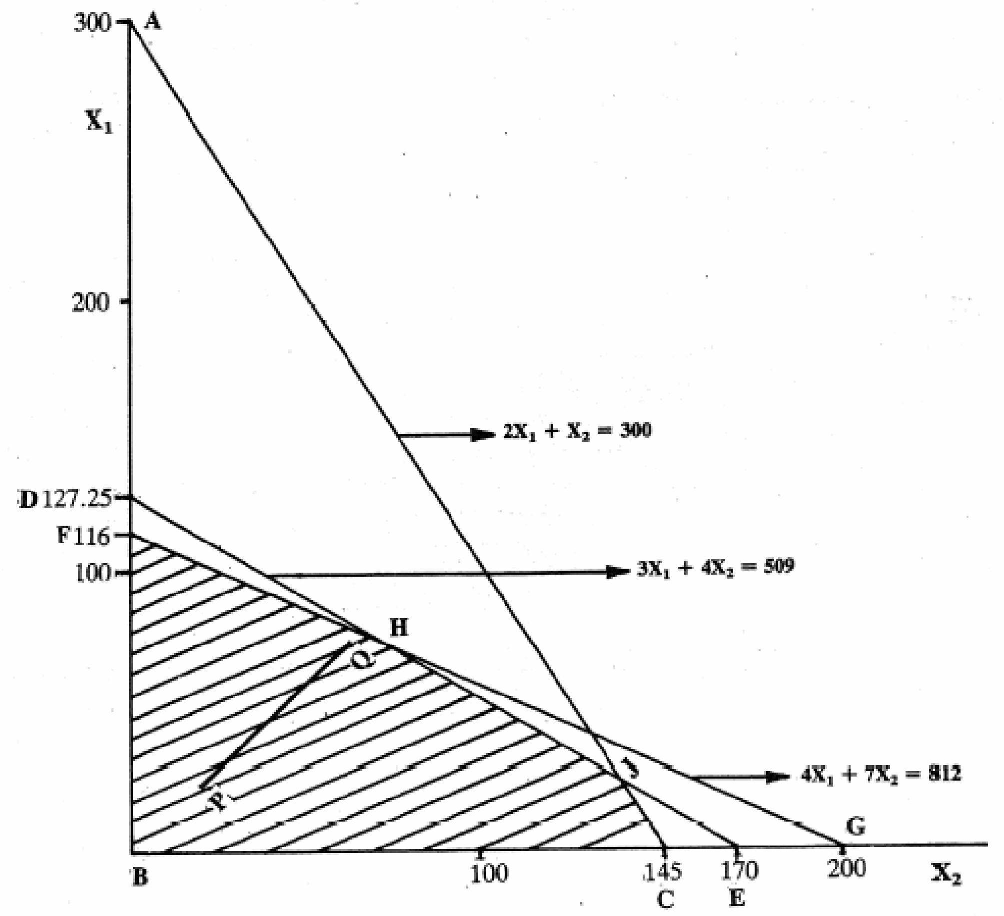

Figure 3.4 : Feasible Region

Feasible solution and feasible region

Any non-negative value of (x_1, x_2) i.e (x_1, x_2) i.e. $x_1 \geq 0,\ x_2 \geq 0$ is a feasible solution of the linear programming problem if it satisfies all the constraints. The collection of all feasible solutions is known as the feasible region.

The feasible region of the linear programming problem under discussion is indicated by the shaded part of Figure 3.4.

Example 4

Consider the linear programming problem formulated in Section 3.3.

Minimise $2x_1 + 1.7x_2$

Subject to :

$$0.15x_1 + 0.10x_2 \geq 1.0$$
$$0.75x_1 + 1.70x_2 \geq 7.5$$
$$1.30x_1 + 1.10x_2 \geq 10.0$$
$$x_1 \geq 0,\ x_2 \geq 0$$

The feasible region of this problem is indicated in Figure 3.5. It may be noted that as the constraints are of "greater than or equal to" type of feasible region is unbounded on one side.

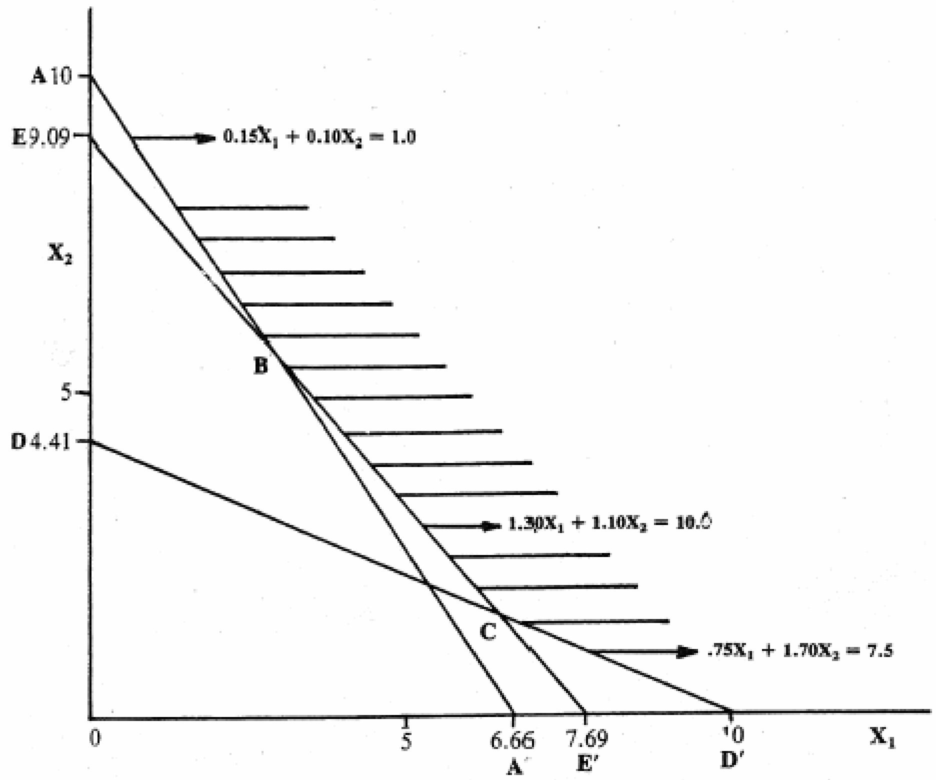

Figure 3.5 : Feasible Region

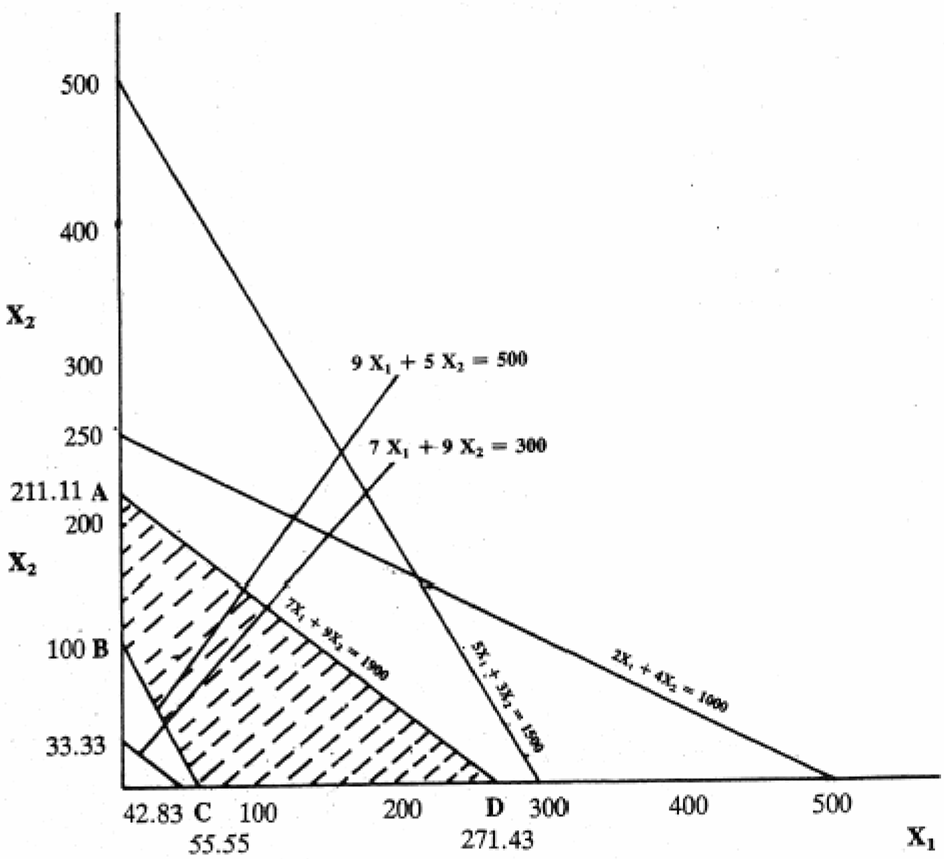

Figure 3.6 : Feasible Region and Redundant Constraints

Example 5

Consider the linear programming problem formulated in Section 3.3.

Maximise $1170x_1 + 1110x_2$

Subject to :

$$9x_1 + 5x_2 \geq 500$$
$$7x_1 + 9x_2 \geq 300$$
$$5x_1 + 3x_2 \leq 1500$$
$$7x_1 + 9x_2 \leq 1900$$
$$2x_1 + 4x_2 \leq 1000$$
$$x_1, \; x_2 \geq 0$$

The feasible region of this linear programming problem is indicated in Figure 3.6.

The feasible region in this case presents another interesting feature. The critical region has been formed by the two constraints.

$$9x_1 + 5x_2 \geq 500$$
$$7x_1 + 9x_2 \leq 1900$$
$$x_1 \geq 0, \; x_2 \geq 0$$

The remaining three constraints although present is not affecting the feasible region in any manner. Such constraints are known as redundant constraints.

Activity 6

Find graphically the feasible region of the linear programming problem given in Activity 2.

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Activity 7

Find graphically the feasible region of the' linear programming problem given in Activity 3.

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Activity 8

Find graphically the feasible region corresponding to the linear programming problem given in Activity 5.

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Activity 9

A company is interested in the analysis of two products which can be made from the idle time of labour, machine and investment possible in this regard. It was found on investigation that the labour requirement for the first and the second products was 2 and 3 units respectively and the total available man hours was 24. Only product 1 required machine hour utilization of one hour per unit and at present only 9 spare machine hours are available, Product 2 requires one unit of a byproduct per unit and the daily availability of the byproduct is .6 units. According to the marketing department the sales potential of product 1 cannot exceed 5 units. In a competitive market, product 1 can be sold at a profit of Rs. 3 and product 2 at a profit of Rs. 5 per unit.

Formulate the problem as a linear programming problem. Determine graphically the feasible region. Identify the redundant constraint.

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3.5 GRAPHICAL SOLUTION

Linear programming with two decision variables can be solved graphically, Although the method is quite simple the principle of solution is based on certain analytical concepts.

Convex Set

A region or a set R is convex if and only if for any two points on the set R the line segment connecting those points lies entirely in R.

We refer to the feasible region presented in Figure 3.4. The points P (x_i = 20, x2 = 25) and Q (x_1 = 60, x_2 = 75) are both feasible solutions of the corresponding linear programming problem. The line joining P and Q belongs entirely in R. Thus the collection of feasible solutions in a linear programming problem form a convex set. In fact, it is a special type of convex set known as convex polygon as this is formed by the intersection of a finite number of closed half planes.

Extreme Point

The extreme point E of a convex set R is a point such that it is not possible to locate two distinct points in or on R so that the line joining them will include E. Extreme points, are also referred to as **vertices** or **corner points**.

We refer to Figure 3.4 for illustration. The point B (origin) is such that it is not 'possible to locate two distinct points in or on the convex set such that B belongs to the line joining them, Thus R is an extreme point of the convex set of feasible solutions. Other extreme points are F, H, J and C.

The following result (Hadley, 1969), (Mittal, 1976) provides the solution of a linear programming problem:

If the maximum or minimum value of a linear function defined over a convex polygon exists, then it must be on one of the extreme points.

We now illustrate the graphical solution of linear programming problems through the following examples.

Example 6

Maximise $50x_1 + 60x_2$

Subject to :

$$2x_1 + x_2 \leq 300$$
$$3x_1 + 4x_2 \leq 509$$
$$4x_1 + 7x_2 \leq 812$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

The feasible region which is a convex polygon is illustrated in Figure 3.7. The extreme points of this convex set are B, F, H, J and C.

The objective function of this problem is $50x_1 + 60x_2$. If we consider M as a parameter the graph $50x_1 + 60x_2 = M$ is a family of parallel straight lines with slope = $-\frac{1}{6}$. Some of these lines will intersect the feasible region and contain many feasible solutions while the others will miss and contain no feasible solution. We wish to find the line of this family that intersects the feasible region and is farthest out from the origin as the problem is to maximise the objective function. The farthest is the line from the origin the greater will be the value of M.

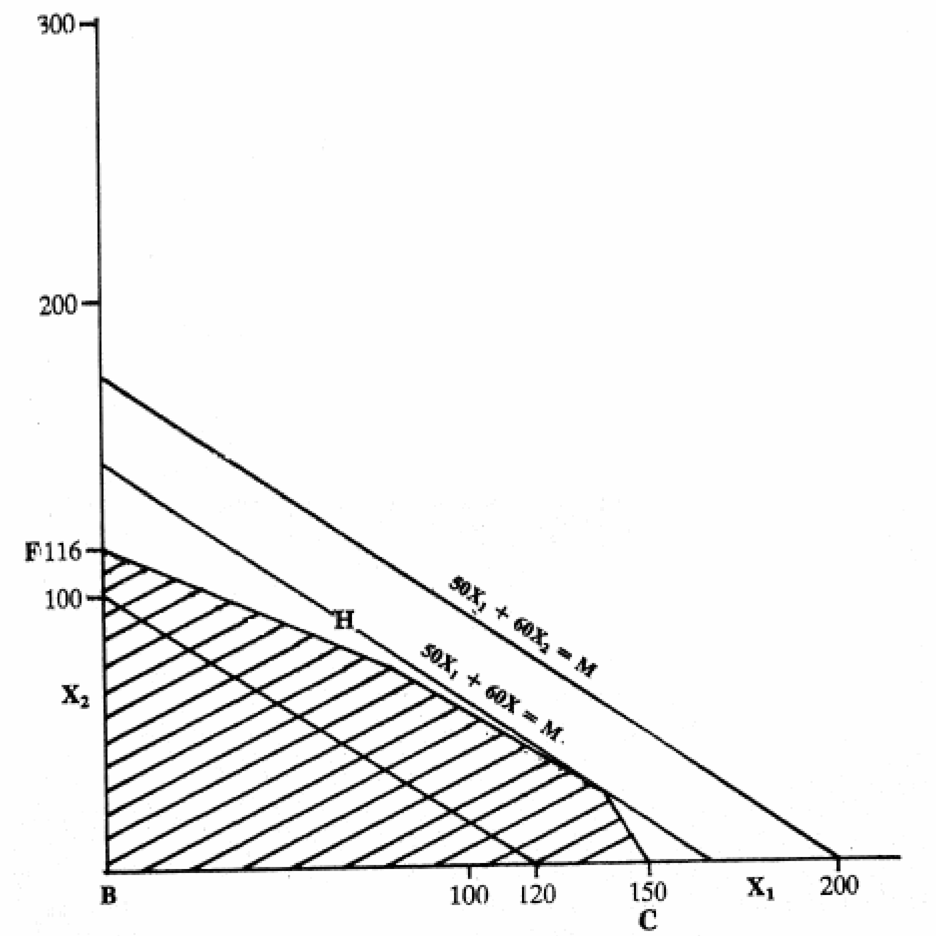

Figure 3.7 : Graphical Solution

In the Figure 3.7 we observe that the line $50x_1 + 60x_2 = M$ passes through the point J. The point J being the intersection of the lines $3x_1 + 4x_2 = 509$ and $4x_1 + 7x_2 = 812$ has the coordinates $x_1 = \frac{691}{5}, x_2 = \frac{118}{5}$. Since J is the only feasible solution on this line the solution is unique. The corresponding value of M is 8326 which is the maximum value of the objective function.

The fact that the objective function is maximised at J can also be ascertained from the values of the objective function at the various extreme points as shown in the Table below.

Table Showing the Computation of Maximum Value of an Objective Function

Extreme Point	Co-ordinates		Profit Function $50x_1 + 60x_2$
B	$x_1 = 0$	$x_2 = 0$	0
C	$x_1 = 150$	$x_2 = 0$	7500
F	$x_1 = 0$	$x_2 = 116$	6960
H	$x_1 = 63$	$x_2 = 80$	7950
J	$x_1 = \frac{691}{5}$	$x_2 = \frac{118}{5}$	8326

Example 7

Minimise $2x_1 + 1.7x_2$

Subject to :

$$0.15x_1 + 0.10x_2 \geqslant 1.0$$
$$0.75x_1 + 1.70x_2 \geqslant 7.5$$
$$1.30x_1 + 1.10x_2 \geqslant 10.0$$
$$x_1 \geqslant 0, \quad x_2 \geqslant 0$$

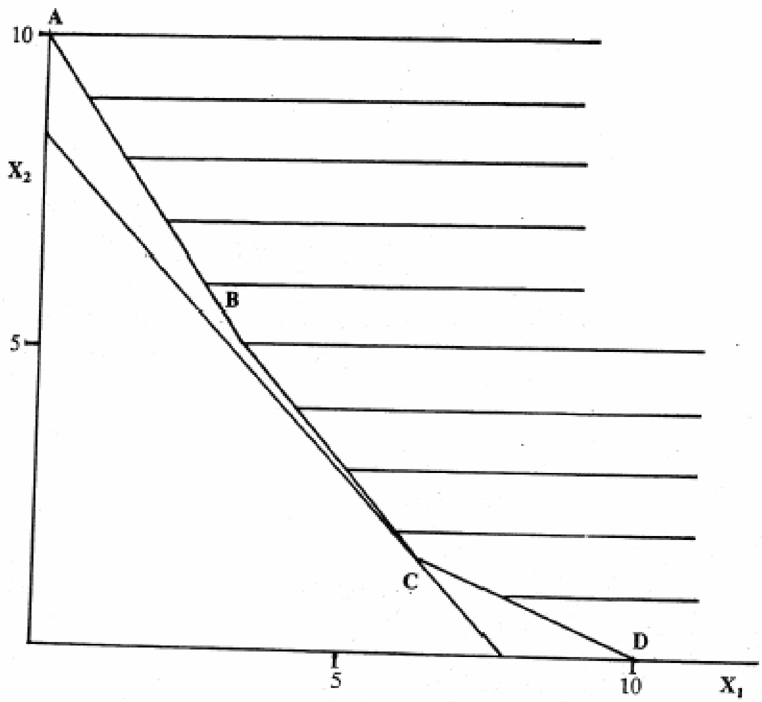

Figure 3.8 : Graphical Solution

Each of the half planes lies above its boundary line (Figure 3.8). The feasible region is infinite at the upper side. It will not be possible to find the maximum in this case hut we are looking for a minimum. Let us introduce a parameter m in $2x_1 + 1.7x_2 = m$ and draw the lines for various values of m. We seek a line in this family of lines that intersects the feasible region and at the same time is as close as possible to the origin. As in the previous example, we compute the value of the objective function at the various extreme points in the following Table.

Table Showing the Computation of Minimum Value of an Objective Function

Table Showing the Computation of Minimum Value of an Objective Function

Extreme Point	Co-ordinates		Objectives Function $2x_1 + 1.7x_2$
A	$x_1 = 0$	$x_2 = 10$	17
B	$x_1 = 2.86$	$x_2 = 5.71$	25.15
C	$x_1 = 6.32$	$x_2 = 1.63$	15.41
D	$x_1 = 10$	$x_2 = 0$	20

Thus the value m is minimum when it passes through the point C. The, optimum values variables are $x_1 = 6.32$ and $x_2 = 1.63$; the minimum value of the objective

Example 8

Maximise $1170x_1 + 1110x_2$

Subject to :

$$9x_1 + 5x_2 \geq 500 \qquad 7x_1 + 9x_2 \leq 1900$$
$$7x_1 + 9x_2 \geq 300 \qquad 2x_1 + 4x_2 \leq 1000$$
$$5x_1 + 3x_2 \leq 1500 \qquad x_1 \geq 0, \;\; x_2 \geq 0$$

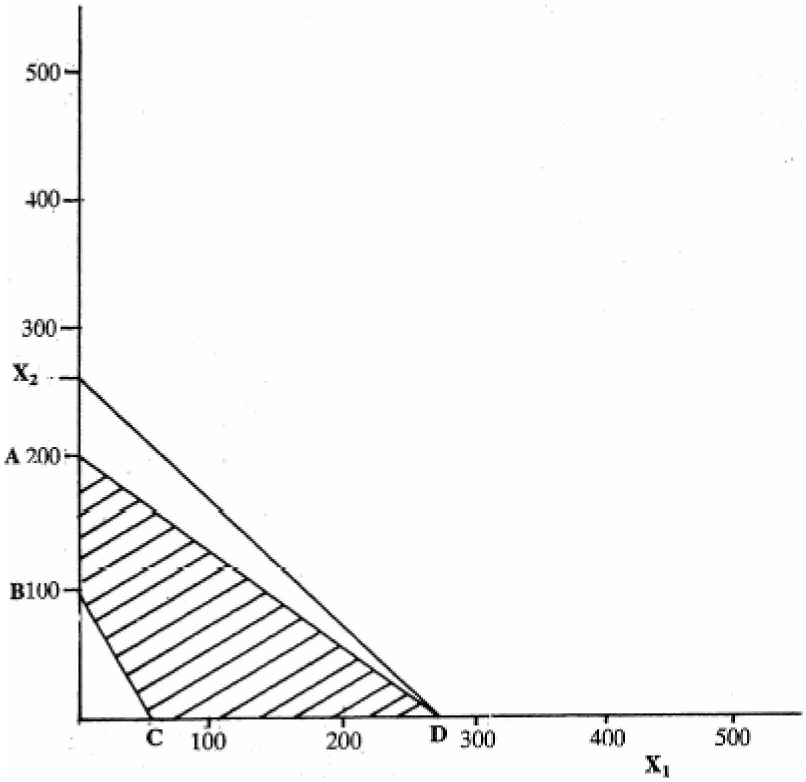

Figure 3.9 : Graphical Solution

As in the previous two examples the maximum value of the objective function is computed in the following Table.

Table Showing the Computation of Maximum Value of an Objective Function

Extreme Point	Co-ordinates		Objectives Function $1170x_1 + 1110x_2$
A	$x_1 = 0$	$x_2 = 211.11$	234332
B	$x_1 = 0$	$x_2 = 100$	111000
C	$x_1 = 55.56$	$x_2 = 0$	65005
D	$x_1 = 271.43$	$x_2 = 0$	317573

Thus the optimum values of the decision variables are $x_1 = 271.43$, $x_2 = 0$ and the maximum value of the objective function is 317573.

Activity 10

Solve graphically the linear programming problem stated in **Activity 2**.

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Activity 11

Solve graphically the linear programming problems stated in Activity 4.

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Activity 12

Solve graphically the linear programming problem stated in Activity 5.

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Activity 13

Solve graphically the linear programming problem stated in Activity 9.

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3.6 MULTIPLE, UNBOUNDED SOLUTION AND INFEASIBLE PROBLEMS

The linear programming problems discussed earlier possessed unique solutions. This was because the objective function passed only through the extreme point located at the intersection of two half planes.

Linear programming problems where the objective function coincides with one of the half planes generated by a constraint will possess multiple solution. The presence of multiple solutions is illustrated through the following example.

Example 9

A company buying scrap metal has two types of scrap metal available to him. The first type of scrap metal has 30% of metal A, 20% of metal B and 50% of metal C by weight. The second scrap has 40% of metal A, 10% of metal B and 30% of metal C. The company requires at least 240 kg. of metal A, 100 kg. of metal B and 290 kg. of metal C. The price per kg. of the two scraps are Rs. 120 and Rs. 160 respectively. 'Determine the optimum quantities of the two scraps to be purchased so that the requirements of the three metals are satisfied at a minimum cost.

Solution

We introduce the decision variables x, and x, indicating the amount of scrap metal to be purchased respectively. Then the problem can be formulated as

$$\text{Minimise } 120x_1 + 160x,$$

Subject to :

$$0.3x_1 + 0.4x_2 \geq 240$$
$$0.2x_1 + 0.1x_2 \geq 100$$
$$0.5x_1 + 0.3x_2 \geq 290$$
$$x_1, x_2 \geq 0$$

Multiplying both sides of the inequalities by 10, the problem become

Minimise $120x_1 + 160x_2$

Subject to :

$$3x_1 + 4x_2 \geq 2400$$
$$2x_1 + x_2 \geq 1000$$
$$5x_1 + 3x_2 \geq 2900$$
$$x_1, x_2 \geq 0$$

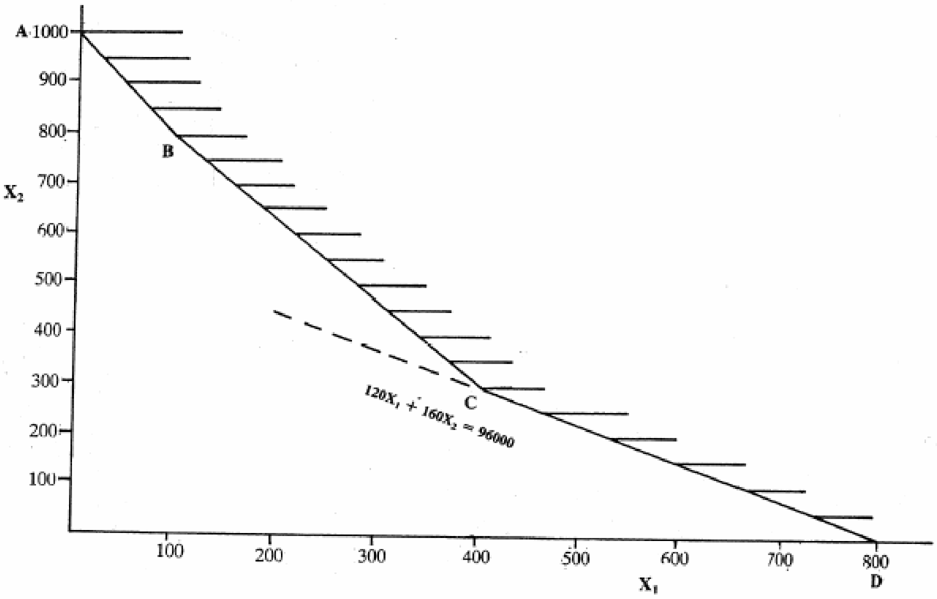

Figure 3.10 : Multiple Solution

The points A, B, C, D are the extreme points of the lower boundary of the convex set of feasible solutions. One of the members of the family of objective functions $120x_1 + 160x_2 = m$ coincides with the line CD with m = 96000. This is indicated by the fact that both the points C with co-ordinates $x_1 = 400$, $x_2 = 300$ and ,D with co-ordinates $x_1 = 800$, $x_2 = 0$ are on the line $120x_1 + 160x_2 = 96000$. Thus, every point on the line CD minimises the value of the objective function and the problem has multiple solutions.

Unbounded Solution

If the feasible region is unbounded it is possible to move the graph of the objective function arbitrarily far out from the origin while passing through the feasible points. In this case no maximum of the objective function exists. The solution of the problem is said to be **unbounded.**

In the previous example the feasible region as shown in Figure 3.10 has no boundary for increasing values of x_1 and x_2. Thus, it is not possible to maximise the objective function in this case and the solution is unbounded.

It may be pointed out that although it is possible to construct linear programming problems with unbounded solutions numerically no linear programming problem formulated from a real life situation can have unbounded solution.. An unbounded solution in this case indicates a wrong formulation or presence of erroneous data.

Infeasible Problem

A linear programming problem is said to be **infeasible** if no feasible solution of the problem exists.

An infeasible linear programming problem with two decision variables can be identified through its graph. This is illustrated in the following example.

Example 10

A company buying scrap metal has two types of scrap available to them. The first type of scrap metal has 20% of metal A, 10% of impurity and 20% of metal by weight. The second type of scrap has 30% of metal A,10% of impurity and 15% of metal B by weight. The company requires at least 120 kg. of metal A, at most 40 kg. of impurity and at least 90 kg. of metal B. The price for the two scraps are Rs. 200 and Rs. 300 per kg. respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the two metals and the restriction on impurity are satisfied at minimum cost.

Solution

Introduce the decision variable x_1 and x2 indicating the amount of scrap metal (in kg.) to be purchased. The problem can be formulated as

Minimise $200x_1 + 300x_2$

Subject to:

$$0.2x_1 + 0.3x_2 \geqslant 120$$
$$0.1x_1 + 0.1x_2 \leqslant 40$$
$$0.5x_1 + 0.15x_2 \geqslant 90$$
$$x_1, \ x_2 \geqslant 0$$

Multiplying both sides of the inequalities by 10, we obtain

$$2x_1 + 3x_2 \geqslant 1200$$
$$x_1 + x_2 \leqslant 400$$
$$2x_1 + 1.5x_2 \geqslant 900$$

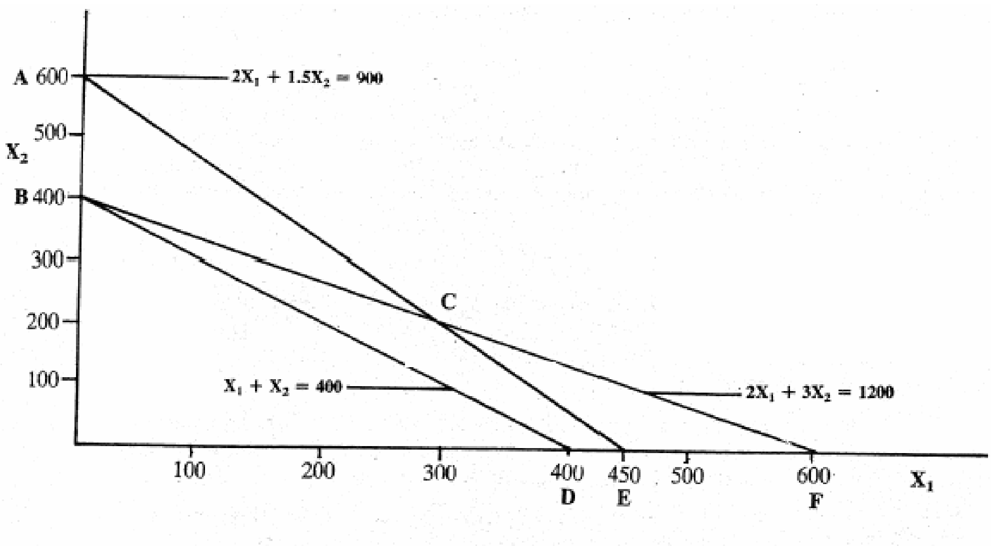

Figure 3.11 : Infeasibility Problem

The region right of the boundary ACF *includes* all *solutions* which satisfy the first And the third constraints. The region located on the left of BD includes all solutions which satisfy the second constraint. Hence, there is no solution satisfying all the three constraints. Thus, the problem is infeasible.

Activity 14

Solve graphically the following linear programming problem

$$\text{Maximise } x_1 + x_2$$

Subject to :

$$-2x_1 + x_2 \leqslant 1$$
$$x_1 \leqslant 2$$
$$x_1 + x_2 \leqslant 3$$
$$x_1,\ x_2 \geqslant 0$$

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Activity 15

Solve graphically the following linear programming problem

$$\text{Maximise } 3x_1 + 2x_2$$

Subject to :

$$x_1 - x_2 \leqslant 1$$
$$x_1 + x_2 \geqslant 3$$
$$x_1,\ x_2 \geqslant 0$$

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Activity 16

Solve graphically the following linear programming problem

$$\text{Maximise } x_1 + x_2$$

Subject to :

$$x_1 + x_2 \leqslant 1$$
$$-3x_1 + x_2 \geqslant 3$$
$$x_1,\ x_2 \geqslant 0$$

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3.7 APPLICATION OF LINEAR PROGRAMMING IN BUSINESS AND INDUSTRY

In the previous sections you have learnt the formulation and solution of a variety of linear programming problems. These include the product mix problem, the problem of preparing a balance diet, the problem of blending in a oil refinery, and a problem of purchasing. You will be familiar with certain other types problems in business and industry which can be formulated and solved by linear programming technique.

Example 11 (Advertising)

The Consumer Product Corporation wishes to plan its advertising strategy. There are 'two magazines under consideration, magazine I and magazine II. Magazine I has a reach of 2000 potential customers per advertisement and magazine II has a reach of 3000 potential customers per advertisement, The cost per advertising is Rs. 6000 and Rs. 9000 in magazines I and II respectively and the firm has a monthly budget of Its, 1 lakh. There is an important requirement that the total reach for the income group under Its 20000 per annum should not exceed 3000 potential customers. The reach of magazine I and II for this income group is 300 and 150 potential customers respectively per advertisement. How many times the company should advertise in the two magazines to maximise the total reach?

Solution

Formulation of the problem

Suppose x_1 is the number of advertisement in magazine I and x_2 is the number of advertisement in magazine II. Then the problem can be formulated as

$$\text{Maximise } 2000x_1 + 3000x_2$$

Subject to :

$$6000x_1 + 9000x_2 \leq 100000$$
$$\text{i.e. } 6x_1 + 9x_2 \leq 100$$
$$300x_1 + 150x_2 \leq 3000$$
$$\text{i.e. } 2x_1 + x_2 \leq 20$$
$$x_1,\ x_2 \geq 0$$

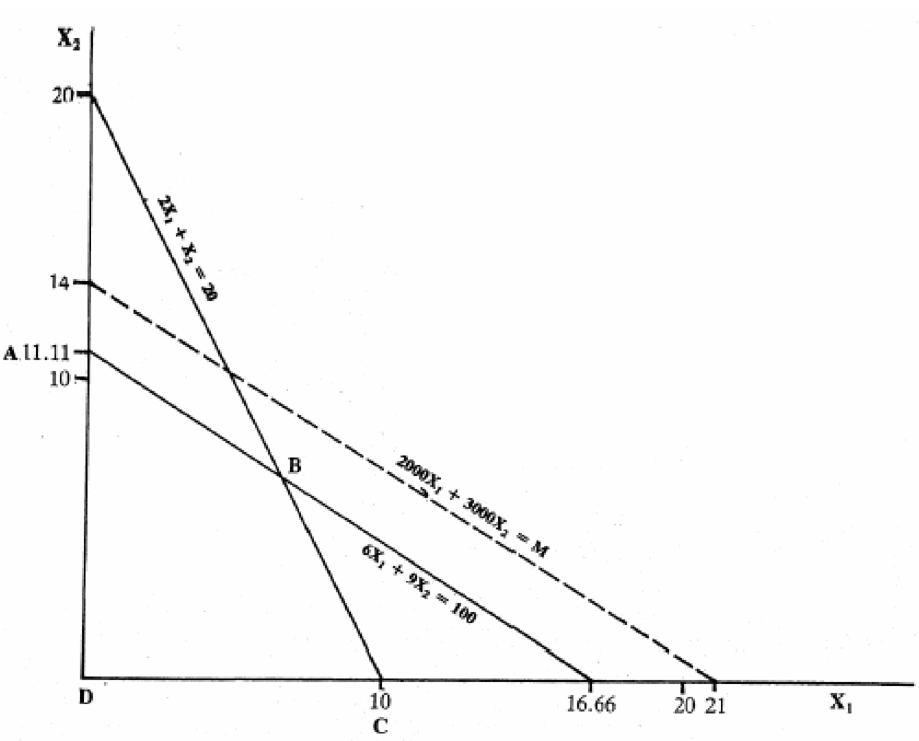

Figure 3.12 : Solution of the Advertising Problem

The feasible region of the problem is indicated by the shaded part ABCD. You may also note that the objective function indicated by the line $2000x_1$ + 3000)($_2$ = M is parallel to the half plane $6x_1 + 9x_2 = 100$. Hence every point on the line AB including the two extreme points A and B are solution of the problem. From the graph it is easy to to observe that the co-ordinate of A is $\left(0, \frac{100}{9}\right)$ and the co-ordinate of B is $\left(\frac{20}{3}, \frac{20}{3}\right)$.

The value of the objective function at both these points is $\frac{100000}{3}$. The problem thus has multiple solutions and any point on the line AB is an optimum solution of the problem.

Example 12 (Cost minimisation)

The final product of a firm has a requirement that it must weigh exactly 150 kg. The two raw materials used in the manufacture of this product are A with a cost of

Rs. 2 per unit and B with a cost of Rs. 8 per unit. Each unit of A weighs 5 kg. and each unit of B weighs 10 kg. At least 14 units of B and no more than 20 units of A must be used. How much of each type of raw material should be used for each unit of the final product if cost is to be minimised?

Solution

We introduce decision variables x_1, x2 indicating the number of units of raw material A and raw material B respectively. Then the problem can be formulated as

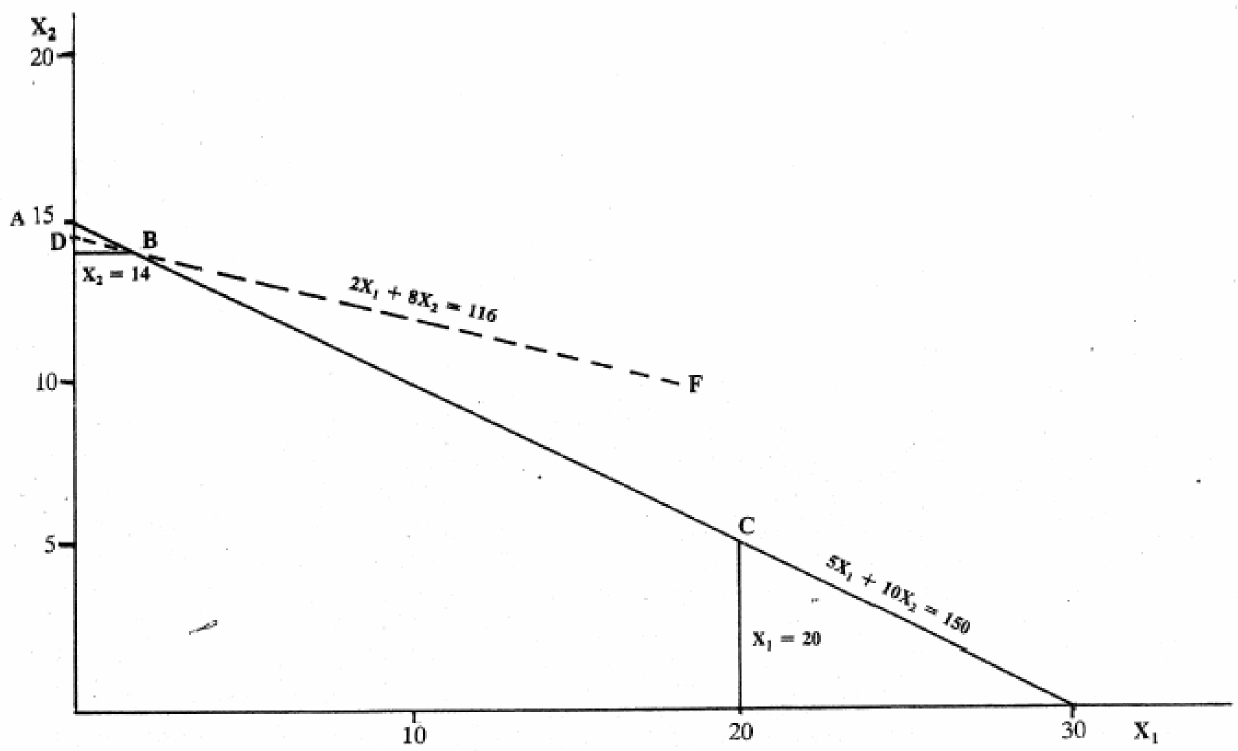

Figure 3.13 : Solution of the Cost Minimisation Problem

The feasible region of the problem is the triangle ADB. The line $2x_1 + 8x_2 = 11$ a passes through the extreme point B. Hence the optimum solution of the problem $_{1,,,}$ given by $x_1 = 2$, $x_2 = 14$ with a minimum cost of Rs. 116

Example 13 (Packaging)

A manufacturer of packing material, manufactures two types of packing tins, round and flat. Major production facilities involved are cutting and joining. The cutting 'department can process 300 round tins or 500 flat tins per hour. The joining department can process 500 round tins or 300 flat tins per hour. If the contribution towards profit for a round tin is the same as that of a flat tin what is the optimum production level?

Solution

Let us introduce decision variables x_i No. of round tins per hour, x_2 = No. of flat tins per hour. Since the contribution towards profit is identical for both the products the objective function can be expressed as $x_1 + x_2$. Hence the problem can be formulated as

Maximise $x_1 + x_2$

Subject to :

$$\frac{1}{300}x_1 + \frac{1}{500}x_2 \leq 1$$

$$\frac{1}{500}x_1 + \frac{1}{300}x_2 \leq 1$$

$$x_1 \geq 0,\quad x_2 \geq 0$$

i.e. $5x_1 + 3x_2 \leq 1500$

$$3x_1 + 5x_2 \leq 1500$$

$$x_1 \geq 0,\quad x_2 \geq 0$$

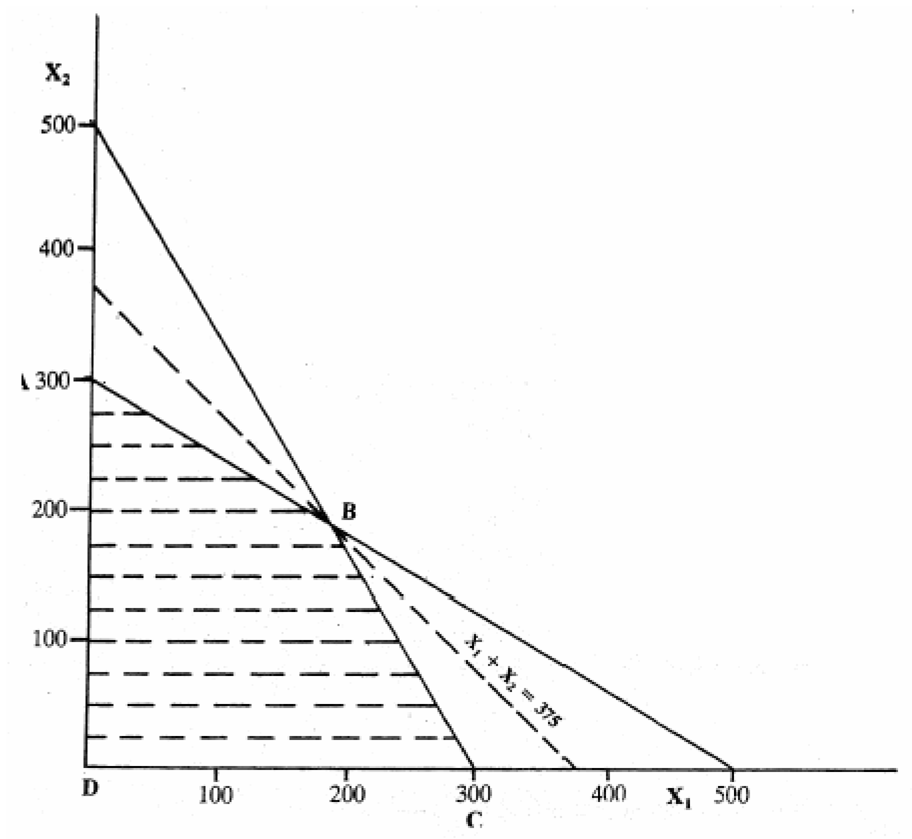

Figure 3.14 : Solution of the Packaging Problem

The feasible region ABCD is indicated by the shade. The coordinate of the extreme point B is $x_1 = \frac{1500}{8}$, $x_2 = \frac{1500}{8}$. A member of the objective function.i.e. $x_1 + x_2 = 375$ passes through this extreme point (dotted line). Hence the optimum solution is $x_1 = \frac{1500}{8}$, $x_2 = \frac{1500}{8}$ with maximum value of the objective function = 375.

Several other problems which can be formulated as two decision variable linear programming problems are given in the self-assessment exercises.

3.8 SUMMARY

Linear programming is a fascinating topic in operations research with wide applications in various problems of management. Regardless of the functional area a linear programming problem has a number of characteristics: We first identify the decision variables which are some economic or physical quantities whose values are of interest to the management. The problem must have a well defined objective function expressed in terms of the decision variables. The objective function may have to be maximised when it expresses the profit or contribution. In case the objective function indicates a cost, it has to be minimised. The decision variables interact with each other through some constraints. These constraints occur due to limited resources, stipulation on quality, technical, legal or a variety of other reasons. The objective function and the constraints are linear functions of the decision variables.

When a problem of management is expressed in terms of the decision variables with appropriate objective function and constraints we say that the problem has been formulated. A linear programming problem with two decision variables can be solved graphically. Any non negative solution which satisfies all the constraints is known as a feasible solution of the problem. The collection of all feasible solutions is known as a feasible region. The feasible region of a linear programming problem is a convex set. The value of the decision variables which maximise or minimise the objective function is located on the extreme point of the convex set formed by the feasible solutions. This point and hence the solution of a linear programming problem with two decision variables can be identified graphically. In some problems, there may be more than one solution. It is also possible that a linear programming problem has no finite solution. Sometimes the problem may be infeasible indicating that no feasible solution of the problem exists. The diverse applicability of linear programming is illustrated in this unit.

3.9 KEY WORDS

Decision Variables are economic or physical quantities whose numerical values indicate the solution of the linear programming problem.

The Objective Function of a linear programming problem is a linear function of the decision variables expressing the objective of the decision maker.

Constraints of a linear programming problem are linear equations or inequalities arising out of practical limitations.

A Closed Half Plane is a linear inequality in two variables which include the value of the variables for which equality is attained.

A Feasible Solution of a linear programming problem is a solution which satisfies all the constraints including the non negativity constraints.

The Feasible Region is the collection of all feasible solutions.

A Redundant Constraint is a constraint which does not affect the feasible region.

A Convex Set is a collection of points such that for any two points on the set, the line joining the points belongs to the set.

A Convex Polygon is a convex set formed by the intersection of a finite number of closed half planes.

An Extreme Point of a convex set is a point such that it is not possible to locate two distinct points in or on the set such that the line joining the latter two points will include the first point.

Multiple Solutions of a linear programming problem are solutions each of which maximise or minimise the objective function.

An Unbounded Solution of a linear programme problem is a solution whose objective function is infinite.

An Infeasible Linear Programming Problem has no feasible solution.

3.10 SELF-ASSESSMENT EXERCISES

1) Laxmi Furniture Mart (LFM) is in the business of manufacturing tables and chairs. In a day, LFM has 40 hours for assembly and 32 hours of finishing work. Manufacturing of a table requires 4 hours in assembly and 2 hours in finishing. A chair requires 2 hours in assembly and 4 hours in finishing. Profitability analysis indicates that every table would contribute Rs. 80, while a chair's contribution is Rs. 55. Because of the long standing business experience, LFM does not find it difficult to sell their products. What should be daily production to maximise the contribution?

2) The India Manufacturing Corporation (IMC) has one plant located on the outskirts of the city. Its production is limited to two products naphtha and urea. The unit contribution for each product has been computed as Rs. 50 per unit of naphtha and Rs. 60 per unit of urea. The time requirements for each product and total time available in each department are as follows :

Department	**Hours Required**		**Available Hours in a Month**
	Naphtha	**Urea**	
1	2	3	1500
2	3	2	1500

In addition, the demand for the products restrict the production to a maximum of 400 units of each of these two products. What should be the daily production schedule so as to maximise the contribution?

3) A poultry farmer feeds his hens with an all-purpose grain and a balanced poultry feed. The all purpose grain costs Rs. 2 per kg. while the balanced feed costs Rs. 3 per kg. A kg. of grain provides 70 calorie units and 25 vitamin units while a kg. of balanced feed provides 75 calorie units and 50 vitamin units. For proper growth and egg production; the hens must be fed with a minimum of 100 calorie units and 75 vitamin units every week. What is the minimum weekly cost food programme and what is the extent of two ingredients in this food programme?

4) A company is making two different types of radio. The profit per unit of the first radio is Rs. 60 while the profit per unit of the second radio is Rs. 25. The first radio require 7 hours of assembly and 12 hours of body work. The second radio requires 9 hours of assembly work and 5 hours body work. The daily hours available for assembly work and body work are 63 hours and 60 hours respectively. Find the daily production schedule which maximises the profit. Give your comments about the solution.

5) Find the solution of the following linear programming problem if possible.

Maximise $3x_1 - 4x_2$

Subject to :

$$x_1 - x_2 \geq 0$$
$$x_2 \leq 6$$
$$x_1 \geq 0, \ x_2 \geq 0$$

6)

Maximise $3x_1 + 2x_2$

Subject to :

$$x_1 + 2x_2 \leq 2$$
$$2x_1 + x_2 \geq 6$$

$x_1 \geq 0, \; x_2 \geq 0$

7) Maximise $4x_1 + 3x_2$

Subject to :

$2x_1 + 3x_2 \leq 6$

$4x_1 + 6x_2 \geq 24$

$x_1, \; x_2 \geq 0$

8) Maximise $3x_1 + 2x_2$

Subject to :

$2x_1 - 3x_2 \geq 0$

$3x_1 + 4x_2 \leq -12$

$x_1, \; x_2 \geq 0$

9) The ABC Company has been a producer of picture tubes for television sets and certain printed circuits for radios. The company has just expanded into full scale production and marketing of AM and AM-FM radios. It has built a new plant that can operate 48 hours per week. Production of an AM radio in the new plant will require 2 hours and production of an AM-FM radio will require 3 hours. Each AM radio will contribute Rs. 40 to profits, while an AM-FM radio will contribute Rs. 80 to profits. The market department, after extensive research, has determined that a maximum of 15 AM radios and 10 AM-FM radios can be sold each week. Determine, using grapical method, the optimal production mix of AM and AM-FM radios that will maximise profits.

3.11 ANSWERS

Activity 1

i) linear, maximised, minimised.

ii) resources, equations, inequalities.

iii) decision variables, maximise, minimise, constraints.

Activity 2

Bottled juice x_1

Canned apple x_2

Maximise $x_1 + 2x_2$

Subject to :	12.
$3x_1 + 2x_2$	0
$x_1 + 2.3x_2$	6.9
$x_1 + 1.4x_2$	4.9

$x_1 \geq 0, \; x_2 \geq 0$

Activity 3

x_1 : Black and white sets

x_2 : Colour sets

Maximise $6x_1 + 15x_2$

Subject to :

$x_1 + x_2 \leq 24$

$5x_1 + 10x_2 \leq 160$

$x_2 \leq 10$

$x_1, \; x_2 \geq 0$

Activity 4

Mill A x_1
Mill B x_2
Minimise $1000x_1 + 800x_2$

Subject to :

$$6x_1 + 2x_2 \geq 12$$
$$2x_1 + 2x_2 \geq 8$$
$$4x_1 + 12x_2 \geq 24$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

Activity 5

Scrap A x_1
Scrap B x_2
Minimise $12.5x_1 + 14.5x_2$

Subject to :

$$x_1 + x_2 \geq 2000$$
$$0.4x_1 + 0.75x_2 \geq 1000$$
$$0.075x_1 + 0.1x_2 \leq 200$$

Activity 9

Product 1 x_1
Product 2 x_2
Maximise $3x_1 + 5x_2$

Subject to :

$$2x_1 + 5x_2 \leq 24$$
$$x_1 \leq 9$$
$$x_2 \leq 6$$
$$x_1 \leq 5$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

The constraint $x_1 \leq 9$ is redundant

Activity 10

Bottled Juice $\frac{161}{90}$

Canned Apple $\frac{20}{9}$

Maximum Profit Rs. $\frac{187}{30}$

Activity 11

$x_1 = 1$, $x_2 = 3$
Minimum Cost Rs. 3400

Activity 12

$x_1 = 1429$
$x_2 = 571$
Minimum Cost Rs. 26142

Activity 13

$x_1 = 3$, $x_2 = 6$
Maximum Profit Rs. 39

Activity 14

The problem has multiple solutions

$$x_1 = 2, \quad x_2 = 1 \quad \text{or} \quad x_1 = \frac{2}{3}, \quad x_2 = \frac{7}{3}$$

Maximum value of the objective function = 3.

Activity 15

The solution is unbounded.

Activity 16

The problem is infeasible.

Self-Assessment Exercises

1) Tables 8, Chairs 4, Maximum Profit Rs, 860.
2) Naphtha 300 units, Urea 300 units, Maximum profit Rs.33000.
3) Balanced feed $1\frac{1}{2}$ units. Minimum Cost Rs. $4\frac{1}{2}$.
4) The problem has multiple solutions

   First type of radio 5, second type of radio 0

   First type of radio 5, second type of radio 73

   Maximum Profit Rs. 300
5) The solution is unbounded
6) The problem is infeasible
7) No feasible solution
8) No feasible solution
9) AM radio 9

   AM-FM radio 10

   Maximum profit Rs. 1160

3.12 FURTHER READINGS

Hadley, G. 1969. *Linear Programming,* Addison Wesley, Reading, Massachusetts : USA.

Mittal, *K.V.*1976. *Optimization Methods in Operations Research and Systems Analysis,* Wiley Eastern Limited New Delhi.

Mustafi, C.K. 1988. *Operations Research Methods and Practice,* Wiley Eastern Limited : New Delhi.