Some Special Expectations:

Mean: Let X be a r.v. whose expectation exists. The mean value μ of X is defined to be $\mu = E(X)$.

Variance: Let X be a random variable with finite mean μ such that $E[(X-\mu)^2]$ is finite. Then the variance of X is defined to be $E[(X-\mu)^2]$.

Notation: σ_X^2 or $Var(X)$ or σ^2

$$Var(X) = \sigma_X^2 = E[(X-\mu)^2]$$

Now,
$$\sigma^2 = E[(X-\mu)^2] = E(X^2 - 2\mu X + \mu^2)$$
$$= E(X^2) + E(-2\mu X) + E(\mu^2)$$
$$= E(X^2) - 2\mu E(X) + \mu^2$$
$$= E(X^2) - 2\mu^2 + \mu^2 \qquad [\because E(X) = \mu]$$
$$= E(X^2) - \mu^2$$
$$= E(X^2) - (E(X))^2$$

This formula frequently affords an easier way of computing the variance of X.

$Var(X) \geq 0 \qquad [\because E((X-\mu)^2)$ is always non-negative$]$

$\Rightarrow E(X^2) - (E(X))^2 \geq 0$

$\Rightarrow E(X^2) \geq (E(X))^2$

Thm: Let X be a r.v. with mean μ and variance σ^2.

Then $Var(aX+b) = a^2 Var(X)$, where a & b are constants.

Proof: $\because E(aX+b) = a\mu + b$. Now use $Var(aX+b) = E\{[(aX+b) - (a\mu

$$= E\{[a(x-\mu)]^2\} = E\{a^2(x-\mu)^2\} = a^2 E\{(x-\mu)^2\}$$

$\therefore Var(ax+b) = a^2 Var(x)$.

Example: $f(x) = \frac{1}{2a}$, $-a < x < a$, zero elsewhere.

find mean and variance.

Soln

$$\mu = E(X) = \int_{-a}^{a} x\left(\frac{1}{2a}\right)dx = 0.$$

$$E(x^2) = \int_{-a}^{a} x^2\left(\frac{1}{2a}\right)dx = \frac{1}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = a^2/3$$

$$Var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = a^2/3 - 0 = a^2/3.$$

Q. The quantity $E[(x-a)^2]$ is minimum, when $a = \mu = E(X)$.

Soln Consider $E[(x-a)^2] = E[\{(x-\mu)+(\mu-a)\}^2]$

$$= E[(x-\mu)^2 + 2(x-\mu)(\mu-a) + (\mu-a)^2]$$

$$= E(x-\mu)^2 + 2(\mu-a)\{E(x)-\mu\} + E(\mu-a)^2$$

$$= E(x-\mu)^2 + (\mu-a)^2 \qquad \text{——— (1)}$$

$\because (\mu-a)^2 \geq 0$

So $E((x-a))^2$ is minimum, when $(\mu-a)^2 = 0$

i.e. $a = \mu \Rightarrow a = E(X)$

Q. If $x^* = \frac{x-\mu}{\sigma}$ is a standard random variable, then

$E(x^*) = 0$ and $Var(x^*) = 1$.

Q. The r.v. X has the p.m.f. $P(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \ldots$, zero elsewhere.

find mean and variance.

Q. Suppose a r.v. X have a pdf $f(x)$, whose graph is symmetric about $x = c$, if the mean of X exists. Then show that

$E(X) = c$

[illegible] $f(x)$ is symmetric about $x = c$

$f(c - x) = f(c + x)$ —— (1)

[To show $E(X) = c$ or $E(X) - c = 0$ or $E(x - c) = 0$]

$E(X - c) = \int_{-\infty}^{\infty} (x - c) f(x)\, dx$

$= \int_{-\infty}^{c} (x - c) f(x)\, dx + \int_{c}^{\infty} (x - c) f(x)\, dx = I_1 + I_2$ —— (2)

$I_1 = \int_{-\infty}^{c} (x - c) f(x)\, dx$

$= \int_{-\infty}^{0} z\, f(z + c)\, dz$

[take $x - c = z \Rightarrow dx = dz$
limits: $x = -\infty \Rightarrow z = -\infty$
$x = c \Rightarrow z = 0$]

$= \int_{\infty}^{0} (-x) f(c - x)(-dx)$

[take $z = -x$
$\Rightarrow dz = -dx$]

$= -\int_{0}^{\infty} x\, f(c - x)\, dx$ —— (3)

$I_2 = \int_{c}^{\infty} (x - c) f(x)\, dx$

$= \int_{0}^{\infty} z\, f(c + z)\, dz$

[take $x - c = z \Rightarrow dx = dz$
limits: $x = \infty \Rightarrow z = \infty$
$x = c \Rightarrow z = 0$]

$= \int_{0}^{\infty} x\, f(c + x)\, dx$ —— (4) [Replacing variable z by x]

[illegible] (1) [illegible] $+ \int_{0}^{\infty} x\, f(c + x)\, dx$

$\Rightarrow E(X - c) = \int_{0}^{\infty} x \{ -f(c - x) + f(c + x) \}\, dx = \int_{0}^{\infty} x \{0\}\, dx = 0$ [using eq (1)]

$\Rightarrow E(X - c) = 0 \Rightarrow E(X) - c = 0 \Rightarrow c = E(X)$

Q. Suppose a r.v. X have pdf $f(x)$, whose graph is symmetric about $x=c$ if the mean of X exists Then show that

$$E(X)=c.$$

Solⁿ

$\because$ graph of $f(x)$ is symmetric about $x=c$

$$\therefore f(c-x) = f(c+x) \quad \text{——— (1)}$$

$\boxed{\text{T.S. } E(X)=c \Leftrightarrow E(X)-c=0 \Rightarrow E(X-c)=0}$

Now, $E(X-c) = \int_{-\infty}^{\infty} (x-c) f(x)\, dx$

$$\Rightarrow E(X-c) = \int_{-\infty}^{c} (x-c) f(x)\, dx + \int_{c}^{\infty} (x-c) f(x)\, dx = I_1 + I_2 \quad \text{——— (2)}$$

Here $I_1 = \int_{-\infty}^{c} (x-c) f(x)\, dx$

$= \int_{-\infty}^{0} z\, f(z+c)\, dz$ $\left[\begin{array}{l}\text{take } x-c=z \Rightarrow dx=dz \\ \text{limits: } x=-\infty \Rightarrow z=-\infty \\ x=c \Rightarrow z=0\end{array}\right]$

$= \int_{\infty}^{0} (-x) f(c-x)(-dx)$ $\left[\begin{array}{l}\text{take } z=-x \\ \Rightarrow dz=-dx\end{array}\right]$

$= -\int_{0}^{\infty} x\, f(c-x)\, dx$ ——— (3)

and $I_2 = \int_{c}^{\infty} (x-c) f(x)\, dx$

$= \int_{0}^{\infty} z\, f(c+z)\, dz$ $\left[\begin{array}{l}\text{take } x-c=z \Rightarrow dx=dz \\ \text{limits } x=\infty \Rightarrow z=\infty \\ x=c \Rightarrow z=0\end{array}\right]$

$= \int_{0}^{\infty} x\, f(c+x)\, dx$. ——— (4) [replacing variable z by x]

From eqⁿ (2), $E(X-c) = -\int_0^{\infty} x\, f(c-x)\, dx + \int_0^{\infty} x\, f(c+x)\, dx$

$\Rightarrow E(X-c) = \int_0^{\infty} x\{-f(c-x)+f(c+x)\}\, dx = \int_0^{\infty} x\{0\}\, dx = 0$ [using eqⁿ (1)]

$\Rightarrow E(X-c)=0 \Rightarrow E(X)-c=0 \Rightarrow c=E(X).$