Chapter - 7 (Cosets and Lagrange's Theorem)

Coset of a Subset in a Group

Let G be a group and H be a subset of G.

For any element $a \in G$, the set $aH = \{ah \mid h \in H\}$ is called the left coset of H in G containing a.

Similarly, the set $Ha = \{ha \mid h \in H\}$ is called the right coset of H in G containing a.

Note: $|aH|$ = no of element in aH

$|Ha|$ = no of elements in Ha

Example: $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$. $H = \{0, 1\}$

$0 \in \mathbb{Z}_4$.

$aH = a * H$. $\cancel{0H}$

$0 +_4 H = \{0 +_4 h \mid h \in H\}$
$= \{0, 1\} = H$

$1 +_4 H = \{1 +_4 h \mid h \in H\} = \{1, 2\}$
$2 +_4 H = \{2 +_4 h \mid h \in H\} = \{2, 3\}$
$3 +_4 H = \{3 +_4 h \mid h \in H\} = \{3, 0\} = \{0, 3\}$

Coset of Subgroup H in Group G

For any $a \in G$,

called the

the set $aH = \{ah \mid h \in H\}$ is called the
left coset of subgroup H in G containing a.

and the set $Ha = \{ha \mid h \in H\}$ is called the
right coset of subgroup H in G containing a.

Example $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$.
$H = \{(1), (12)\}$.

find all left cosets of H in G.

$(12) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, $(132) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

$(1) \in G$

① $(1)H = \{(1)\phi \mid \phi \in H\} = \{(1), (12)\}$
$= H$.

② $(12)H = \{(12)\phi \mid \phi \in H\} = \{(12)(1), (12)(12)\}$
$= \{(12), (1)\} = H$.

③ $(13)H = \{(13)\phi \mid \phi \in H\} = \{(13)(1), (13)(12)\}$
$= \{(13), (123)\}$

④ $(23)H = \{(23)\phi \mid \phi \in H\} = \{(23)(1), (23)(12)\}$
$= \{(23), (132)\}$

$(123) = (13)(12)$

$(23)(12)$
$= (23)(21)$
$= (213) = (132)$
$= (321)$

⑤ $(123)H = \{(123)\phi \mid \phi \in H\}$
$= \{(123)(1), (123)(12)\}$
$= \{(123), (13)\}$
$= \{(13), (123)\}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

ex $(132)H$ $\quad (132)(12) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

$$\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \end{pmatrix}$$

(6) $(132)H$ $\qquad (1\,3)(\cancel{2}) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

$= \{(23), (132)\}$

$aH = Ha$ may or may not equal.

$H(132) = \{\phi(132) \mid \phi \in H\}$

$= \{(1)(132), (12)(132)\}$

$= \{(132), (13)\}$

$\therefore (132)H \neq H(132)$

Properties of Cosets:

Let H be a subgroup of G and let $a, b \in G$

Then, (i) $a \in aH$

(ii) $aH = H$ iff $a \in H$

(iii) $aH = bH$ or $aH \cap bH = \phi$

(iv) $aH = bH$ iff $a^{-1}b \in H$

(v) $|aH| = |bH|$

(vi) $aH = Ha$ iff $H = aHa^{-1}$

(vii) aH is a subgroup of G if and only if $a \in H$

Proof: (i) To $a \in aH$

$\because H$ is a subgroup

$\therefore e \in H$

Now $aH = \{ah \mid h \in H\}$

$\because e \in H \Rightarrow ae \in aH$

$\Rightarrow a \in aH.$

(ii) First let $aH = H$.

$\therefore a \in aH$ (from part (i))

$\Rightarrow a \in H$ ($\because aH = H$)

Conversely ($\Leftarrow$) Let $a \in H$.

TS $aH = H$

we will show that $aH \subseteq H$ & $H \subseteq aH$.

$aH = \{ah \mid h \in H\}$

$\because H$ is a subgroup

$\therefore ah \in H \; \forall \; a, h \in H$ (closedness)

$\therefore aH = \{ah \mid h \in H\} \subseteq H$

$\Rightarrow aH \subseteq H$ ——— (1)

Let $h \in H$ be an arbitrary element of H.

$\because a \in H, h \in H \Rightarrow a^{-1}h \in H$ [$\because H$ is a subgroup]

Now $h = eh = aa^{-1}h = a(a^{-1}h)$

$\Rightarrow h \in aH$

$\boxed{h \in H \Rightarrow h \in aH \Rightarrow H \subseteq aH \quad \begin{array}{l} h = ah_1 \\ h_1 \in H \end{array}}$

$\therefore H \subseteq aH$ ——— (2)

$\boxed{H \subseteq aH}$

from eq^n (1) & (2), $aH = H$.

(iii) $aH = bH$ or $aH \cap bH = \phi$

Suppose $aH \cap bH \neq \phi$

let $x \in aH \cap bH$.

$\Rightarrow x \in aH$ and $x \in bH$.

$\Rightarrow x = ah_1$ and $x = bh_2$ where $h_1, h_2 \in H$

Now, $a = xh_1^{-1} = bh_2h_1^{-1}$

and $aH = b\underbrace{h_2h_1^{-1}H}$

$\Rightarrow aH = bH$.

$\left[\because h_2h_1^{-1} \in H \right.$ and using Part (ii) $\left. a \in H \Leftrightarrow aH = H. \right]$

(iv) $aH = bH$ iff $a^{-1}b \in H$.

first let $a^{-1}b \in H$

$\Rightarrow (a^{-1}b)H = H$ (from Property (ii))

$\Rightarrow a^{-1}bH = H$

$\Rightarrow a(a^{-1}bH) = aH$

$\Rightarrow (aa^{-1})bH = aH \Rightarrow (e)\, bH = aH$

$\Rightarrow bH = aH \Rightarrow aH = bH$

Conversely ($\Leftarrow$) let $aH = bH$.

$\Rightarrow a^{-1}(aH) = a^{-1}bH$

$\Rightarrow H = (a^{-1}b)H$.

$\Rightarrow (a^{-1}b)H = H$.

$\Rightarrow a^{-1}b \in H$. (Property (ii))

TS ...

(v) $\underline{\underline{TS}}$ $|aH| = |bH|$

we need to find a one - one correspondence between sets aH and bH.

$\phi : aH \to bH$

$\phi(ah) = bh. \quad \forall h \in H.$

$\underline{\underline{TS}}$ ϕ is one - one & onto.

(vi) $aH = Ha$ iff $H = aHa^{-1}$

$aH = Ha \Leftrightarrow aHa^{-1} = Haa^{-1}$

$\Leftrightarrow aHa^{-1} = He$

$\Leftrightarrow aHa^{-1} = H$

(vii) $\underline{\underline{TS}}$ aH is a subgroup of G iff $a \in H$.

Let aH be the subgroup of G

$\underline{\underline{TS}}$ $a \in H$.

aH is the subgroup of $G \Rightarrow e \in aH$

and $e \in eH$.

$\therefore e \in aH \cap eH \Rightarrow aH \cap eH \neq \phi$

$\Rightarrow aH = eH$

$\Rightarrow aH = H.$ (Property (ii))

$\Rightarrow a \in H.$

Conversely ($\Leftarrow$) let $a \in H$.

$a \in H \Rightarrow aH = H$.
$\Rightarrow aH$ is a subgroup of G.

Centralizer of a

$$C(a) = \{x \in G \mid ax = xa\}$$

chapter 3

Q. 33 $Z(G) = \{x \in G \mid ax = xa \ \forall a \in G\}$

$Z(G) = \bigcap C(a)$

$x \in Z(G) \Rightarrow x \in \bigcap C(a)$

$x \in \bigcap C(a) \Rightarrow x \in Z(G)$

Q. 38 $\quad a \neq b \quad \underline{\underline{TS}} \quad a^2 \neq b^2$ or $a^3 \neq b^3$

Suppose if, $a^2 = b^2$ and $a^3 = b^3$

$\Rightarrow a^2 a = b^2 b$ (left cancellation)

$\Rightarrow a = b$.

$\rightarrow\leftarrow$.

Lagrange's Theorem:

order of a subgroup divides order of Group.
or.

If G is a finite group and H is a subgroup of G,

then $|H|$ divides $|G|$. Moreover, the number of

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distinct left (or right) cosets of H in G is $\frac{|G|}{|H|}$.

Proof: $\because$ G is finite group, there are finite no. of left cosets of H in G.

Let $a_1H, a_2H, \dots, a_rH$ be the distinct left cosets of H in G.

Now, for each $a \in G$, we have $aH = a_iH$ for some $1 \le i \le r$

$\Rightarrow$ $a \in a_iH$ for some $1 \le i \le r$.

$\therefore$ $G \subseteq a_1H \cup a_2H \cup \dots \cup a_rH$. —— (1)

Now clearly, $a_iH \subseteq G$ $\forall$ $1 \le i \le r$.

$\therefore$ $a_1H \cup a_2H \cup \dots a_rH \subseteq G$ —— (2)

$a_i \in G$, H subgroup
$a_iH = \{a_ih \mid h \in H\} \subseteq G$. $h \in G$

From eq^n (1) & (2),

$G = a_1H \cup a_2H \cup \dots a_rH$ —— (3)

Now $a_1H, a_2H, \dots, a_rH$ are distinct left cosets of H in G.

$\therefore$ $a_iH \cap a_jH = \phi$ $\forall$ $1 \le i, j \le r$.

Thus, $|G| = |a_1H| + |a_2H| + \dots + |a_rH|$

$\Rightarrow |G| = \underbrace{|H| + |H| + \dots + |H|}_{r \text{ times}}$ $[\because |aH| = |H|]$

$|aH| = |bH|$
$\Rightarrow |aH| = |eH| \dots$

$\underleftrightarrow{\quad r \text{ times} \quad}$

$|aH| = |bH|$
$\Rightarrow |aH| = |eH| = |H|$

$\Rightarrow |G| = r|H|$

$\Rightarrow |H|$ divides $|G|$
and $r = |G|/|H|$

$\Rightarrow$ No. of distinct left cosets of H in G
are $|G|/|H|$.

① G is a Group of order 6.

then G can have subgroups of order 1, 2, 3 and 6.

G can have proper subgroups of order 2 and 3.

Q. If $a \in G$, then $|a|$ divides $|G|$.

Sol: $\because a \in G \Rightarrow a^2 \in G, a^3 \in G, \ldots$

$\therefore \langle a \rangle \subseteq G$ & $\langle a \rangle$ is a subgroup of G.

$\Rightarrow |\langle a \rangle|$ divides $|G|$

$\Rightarrow |a|$ divides $|G|$.