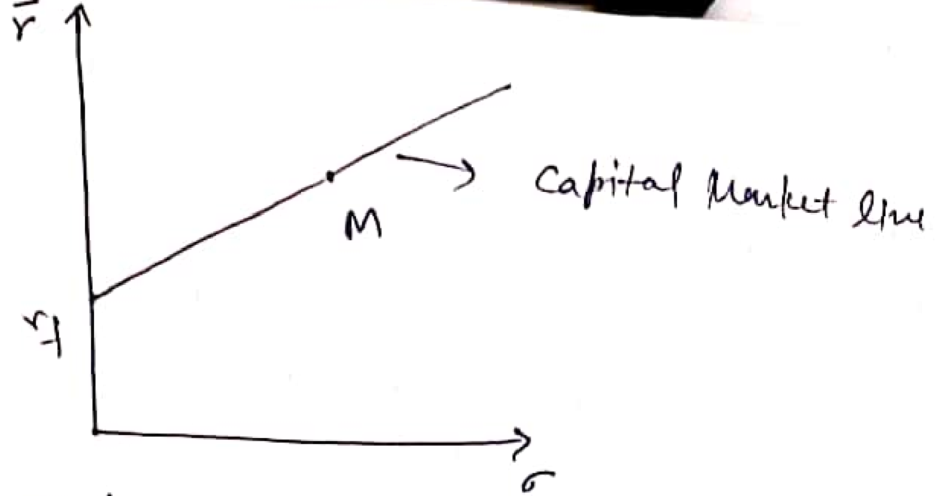
Ch-7 The Capital Asset Pricing model.

This chapter concentrates on the pricing issue. It deduces the correct price of a risky asset within the framework of the mean-variance setting.

Market equilibrium:- Suppose everyone is mean-variance optimizer. Assume there is a unique risk-free rate of borrowing and lending available to all and there are no transaction cost. Then from One-fund theorem everyone will purchase a single fund of risky asset, that fund is called market Portfolio. The weight of an asset in market portfolio is equal to the proportion of that asset's total capital value to the total market capital ~~cot~~ value. Then weights are termed capitalization weights.

The Capital Market line!-

The efficient set consisting of market portfolio and risk-free assets traces a straight line on the $\bar{r}-\sigma$ diagram, the line is called Capital Market line or Pricing line.



In mathematical terms capital market line is

$$\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M} \cdot \sigma$$

where $\bar{r}_M$ and σ_M are the expected value and standard deviation of market rate of return and $\bar{r}$ & σ are the expected value and standard deviation of the rate of return of an arbitrary efficient set.

Slope of Market line $= \dfrac{\bar{r}_M - r_f}{\sigma_M}$

Example ① Suppose risk-free rate is 6%. and market portfolio of risky asset has expected return 12%. and a standard deviation of 15%.. Then the equation of capital market line is

$$\bar{r} = 0.06 + \frac{0.12 - 0.06}{0.15} \sigma$$

Pricing Model

The Capital asset pricing model (CAPM):- If the market portfolio M is efficient, the expected return $\bar{r}_i$ of any asset i satisfies.

$$\bar{r}_i - r_f = \beta_i (\bar{r}_M - r_f), \text{ where}$$

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

Proof:- for any α, consider the portfolio consisting of asset i with weight α and market portfolio with weight $(1-\alpha)$. Then expected rate of return of this portfolio is

$$\bar{r}_\alpha = \alpha \bar{r}_i + (1-\alpha) \bar{r}_M$$

and standard deviation is

$$\sigma_\alpha = \left[\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2\right]^{1/2}$$

As α varies, these values traces out a curve in the $\bar{r}-\sigma$ diagram. and it cannot cross the capital market line, as if it did, the portfolio corresponding to a point above capital market line would violate the definition of capital market line.

Hence as α passes through zero, the curve must be tangent to the capital market line at M.

tangent of the curve $= \left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} =$ tangent of line at M

$= \frac{r_M - r_f}{\sigma_M}$ —①

$$\therefore \frac{d\bar{r}_\alpha}{d\sigma_\alpha} = \frac{\frac{d\bar{r}_\alpha}{d\alpha}}{\frac{d\sigma_\alpha}{d\alpha}}$$

$$\bar{r}_\alpha = \alpha \bar{r}_i + (1-\alpha)\bar{r}_M$$

$$\therefore \frac{d\bar{r}_\alpha}{d\alpha} = \bar{r}_i - \bar{r}_M$$

and as $\sigma_\alpha = \left[\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2\right]^{1/2}$

$$\therefore \frac{d\sigma_\alpha}{d\alpha} = \frac{1}{2\sigma_\alpha}\left[2\alpha\sigma_i^2 + 2(1-2\alpha)\sigma_{iM} + 2(\alpha-1)\sigma_M^2\right]$$

Thus $\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$

then $\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}$

Put the value in eq. ①.

$$\Rightarrow \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{r_M - r_f}{\sigma_M}$$

$$\Rightarrow r_i = r_f + \left(\frac{\bar{r}_M - r_f}{\sigma_M^2}\right)\sigma_{iM}$$

$$= r_f + \beta_i \cdot (\bar{r}_M - r_f) \quad \text{where } \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

CAPM model:-

$$\bar{r}_i - r_f = \beta_i (\bar{r}_M - r_f)$$

where $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$

* Def^n:- β_i is referred as beta of an asset.
$\bar{r}_i - r_f$ is termed as expected excess rate of return
$\bar{r}_M - r_f$ is expected excess rate of return of market portfolio.

Note:- ① Expected excess rate of return is directly proportional to expected excess rate of return of market portfolio.

② Properties of Beta :-

1.) Expected excess rate of return is higher as β is higher.

2.) β is higher as the covariance between an asset and market portfolio is higher.

3) If the asset is uncorrelated with the market, than $\beta = 0$, then By CAPM $\bar{r}_i = r_f$, even if the asset is very risky.

4) If the asset is negatively correlated with market, then $\beta < 0$, then By CAPM, $\bar{r}_i < r_f$ (even with high risk). these type of asset reduce the overall portfolio risk when combined with market.

5) Aggressive companies are supposed to have higher betas

CAPM as pricing model:-

Suppose that an asset is purchased at price P and later sold at price Q. The rate of return is $r = \frac{Q-P}{P}$. Here P is known & Q is random.

then $\bar{r} = \frac{\bar{Q}-P}{P}$. Putting this in CAPM formula,

$$\frac{\bar{Q}-P}{P} = r_f + \beta(\bar{r}_m - r_f)$$

$$\Rightarrow \quad P = \frac{\bar{Q}}{1+r_f+\beta(\bar{r}_m - r_f)}$$

Ex:- Suppose there is an oil venture with $\beta = 0.6$, and expected payoff \$1,000. The risk free rate is 10%. and the expected ~~value~~ return on market portfolio is 0.17. what is the value of this share of oil venture.

$\Rightarrow$ Using CAPM, $r_f = 0.1$, $\bar{r}_M = 0.17$, $\beta = 0.6$

$\bar{Q} = 1{,}000$

$$P = \frac{1000}{1.1+0.6(0.17-0.1)} = 876.$$

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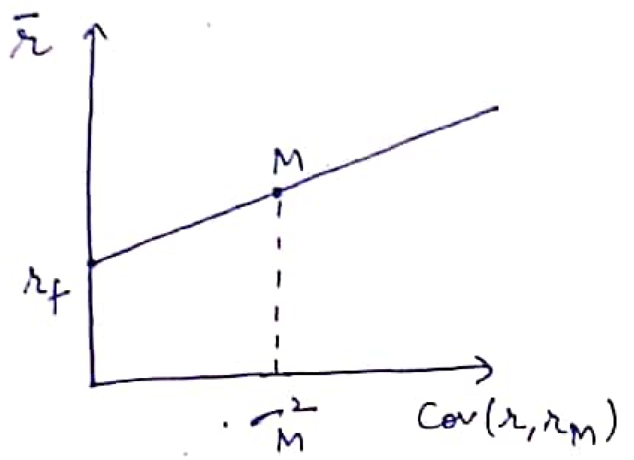
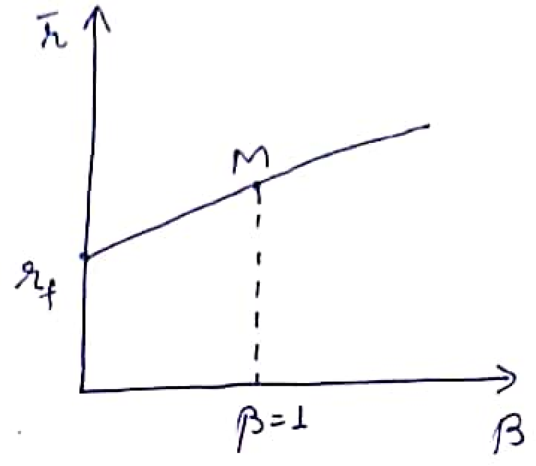
The security market line:-

The CAPM formula can be expressed in graphical form by regarding the formula as linear relationship. This relationship is termed as security market line.





Systematic risk:

Write the random rate of return of asset as

$$r_i = r_f + \beta_i (r_M - r_f) + \epsilon_i \quad — \quad (*)$$

take expected value of (*)

then $\bar{r}_i = r_f + \beta_i (\bar{r}_M - r_f) + E(\epsilon_i)$

Using CAPM, $E(\epsilon_i) = 0$.

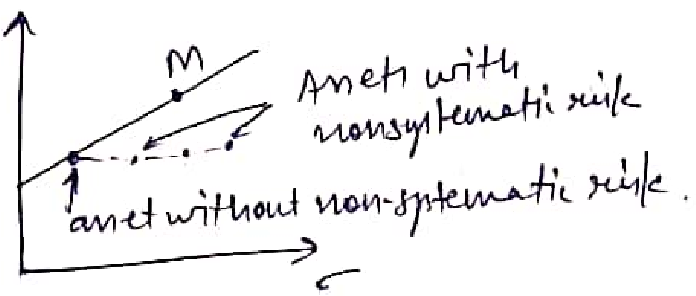
Using * and CAPM, we find $Cov(\epsilon_i, r_M) = 0$.

$$\therefore \quad \sigma_i^2 = \beta_i^2 \sigma_M^2 + var(\epsilon_i)$$

The first part, $\beta_i^2 \sigma_M^2$ is termed as systematic risk and $var(\epsilon_i)$ is termed as nonsystematic, idiosyncratic or specific risk.

Systematic risk cannot by reduced by diversification, because every asset with non-zero beta contains this risk. Whereas, Nonsystematic risk can be reduced by diversification.

- An asset on the capital market line with value of β have standard deviation $\beta\sigma_M$. It has only systematic risk.
- If an asset have nonsystematic risk, they will not fall on the capital market line, Indeed as the nonsystematic risk increases, the points on $\bar{r}$-σ plane representing these assets drift to the right.



Jensen Index:- When we replace expected rate of returns by measured average returns, we add an error term J in CAPM formula, called Jensen's index.

$$\hat{r} - r_f = J + \beta(\hat{r}_M - r_f)$$

where $\hat{r}$ is measured average return.

- J measures how much the performance of an asset deviated from theoretical value of zero. (ABC)

Sharpe index:- It is the slope between the risk-free point and the ABC point on $\bar{r}$-σ diagram.

Question 1) $\bar{r}_M = 0.23$ $\quad r_f = 0.07$ $\quad \sigma_M = 0.32$

(a) $$\bar{r} = 0.07 + \frac{0.23 - 0.07}{0.32} \cdot \sigma \quad -①$$

(b) (i) $\bar{r} = 0.39$

putting values in ①, we find σ.

$\sigma = 0.64$

(ii) Make a portfolio of risk free asset and market portfolio with weights w & $(1-w)$ resp. such that the expected rate of return is 39%.

$\Rightarrow$ $w \times 0.07 + (1-w) \times 0.23 = 0.39$

$\Rightarrow$ $w = -1$

$\Rightarrow$ Borrow \$1,000 at risk-free rate and invest \$2,000 in the market.

(c) Let w_1 be the weight of risk free asset

w_2 ——— market portfolio.

then $w_1 = \frac{3}{10}$, $w_2 = 7/10$. Then expected rate of return of this portfolio

$$\bar{r} = \frac{3}{10} \times 0.07 + \frac{7}{10} \times 0.23$$

$$= 0.021 + 0.161 = 0.182 = 18.2\%.$$

Expected money = $1000(1 + 0.182)$

$= \$1182$

Q2 Here Market portfolio is formed by taking equal weights of assets A and B, i.e. $\frac{1}{2}$.

(a) $$\sigma_M^2 = \frac{1}{2}\times\frac{1}{2}\sigma_A^2 + \frac{1}{2}\times\frac{1}{2}\sigma_B^2 + 2\times\frac{1}{2}\times\frac{1}{2}\sigma_{AB}$$
$$= \frac{1}{4}(\sigma_A^2 + 2\sigma_{AB} + \sigma_B^2)$$

and, $$\beta_A = \frac{\sigma_{AM}}{\sigma_{M^2}}$$

$$\sigma_{AM} = E[(A-\bar{A})(M-\bar{M})]$$
$$= E\left[(A-\bar{A})\left(\frac{1}{2}(A+B) - \frac{1}{2}(\bar{A}+\bar{B})\right)\right]$$
$$= \frac{1}{2}E[(A-\bar{A})(A-\bar{A}+B-\bar{B})]$$
$$= \frac{1}{2}\left(E[(A-\bar{A})(A-\bar{A})] + E[(A-\bar{A})(B-\bar{B})]\right)$$
$$\sigma_{AM} = \frac{1}{2}\cdot(\sigma_A^2 + \sigma_{A,B})$$

$$\Rightarrow \quad \beta_A = \frac{\sigma_A^2 + \sigma_{AB}}{2\sigma_M^2}$$

Ily, $$\beta_B = \frac{\sigma_B^2 + \sigma_{A,B}}{2\sigma_M^2}$$

(b) $$\bar{r}_A = 0.10 + \frac{5}{4}(0.18 - 0.10) = 0.20$$

$$\bar{r}_B = 0.10 + \frac{3}{4}(0.18 - 0.10) = 0.16$$