15th September 2020

# Continuous functions

cont at $c \in A$, $f : A \to \mathbb{R}$

Def → $\varepsilon$-$\delta$ def

$\boxed{5.1.1}$ $\lim_{x \to c} f(x) = f(c)$ ⟷ $f(c)$

$\boxed{5.1.2}$ $f(V_\delta(c) \cap A) \subseteq V_\varepsilon(f(c))$

seq criteria for $L$ to be limit at $c$

$(x_n) \to c$, $x_n \neq c$

then $f(x_n) \to L$.

Divergence Criteria ← $x_n \to c$ but $f(x_n) \not\to L$

**Theorem 5.1.3** seq Criteria for continuity

$f : A \to \mathbb{R}$ is cts at $c \in A$

iff

$\forall$ seq $(x_n)$ in $A$ that conv. to $c$

the seq $(f(x_n))$ converges to $f(c)$

Proof → ~~$0 < |x_n - c|$~~ ; $L \to f(c)$

Hint

HW

**Thm 5.1.4** Discontinuity criterion

Let $A \subseteq \mathbb{R}$ & $f : A \to \mathbb{R}$ & $c \in A$

Then $f$ is disc cts at $c$ iff $\exists$ a seq $(x_n)$ in $A$ such that $(x_n) \to c$ but $f(x_n)$ does not converge to $f(c)$

**Def 5.1.5** Continuity on a set

$f$ is cts at $c$

Let $f : A \to \mathbb{R}$; $B \subseteq A$. $f$ is s.t.b cts on $B$ if $f$ is cts at each point of $B$.

**Examples 5.1.6** (a) Constant function is cts. on $\mathbb{R}$

Let $f(x) = b$ $\forall x \in \mathbb{R}$ $\quad (f : \mathbb{R} \to \mathbb{R})$

$f$ is cts. on $\mathbb{R}$

Let $c \in \mathbb{R}$ be arbitrary

$\lim_{x \to c} f(x) = b$ & $f(c) = b$

$\therefore \lim_{x \to c} f(x) = f(c)$ ⟶ ($\varepsilon$-$\delta$ def)

$\Rightarrow f$ is cts at $c$

As $c \in \mathbb{R}$ is arbitrary $\therefore f$ is cts on $\mathbb{R}$

(b) Identity function let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = x$, $x \in \mathbb{R}$

Then $g$ is cts (on $\mathbb{R}$)

Let $c \in \mathbb{R}$ be arbitrary

$\lim_{x \to c} g(x) = \lim_{x \to c} (x) = c$ & $g(c) = c$

$\therefore \lim_{x \to c} g(x) = g(c)$

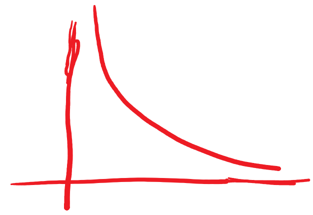
$\Rightarrow g$ is cts at $c \in \mathbb{R}$

As $c \in \mathbb{R}$ is arbitrary $\therefore g$ is cts on $\mathbb{R}$

(c) $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2 \ \forall x \in \mathbb{R}$

$h$ is cts on $\mathbb{R} \longrightarrow$ (Show HW!)

(d) $\phi(x) = 1/x$ is cts on $A = \{x \in \mathbb{R} \mid x > 0\}$



Let $c \in A$. Then $c > 0$

$\lim_{x \to c} \phi(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ ; $\phi(c) = \frac{1}{c}$

$\therefore \lim_{x \to c} \phi(x) = \phi(c)$

$\phi$ is cts at $c \in A$ As $c \in A$ is arbitrary $\Rightarrow \phi$ is cts on $A$

(e) $\phi(x) = 1/x$ is discts at $0$

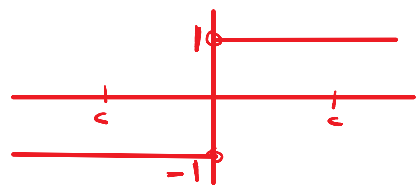
① $\phi(0)$ not def $\Rightarrow \phi$ is discts at $0$

or ② $\lim_{x \to 0} \phi(x)$ d.n.e $\Rightarrow \phi$ is discts at $0$

$\to \phi(0)$ not def

$\to \lim_{x \to 0} \phi(x)$ d.n.e

(f) ① sgn fn is discts at $0$

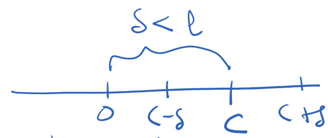


$\lim_{x \to 0} \operatorname{sgn} x$ d.n.e

$\therefore \operatorname{sgn} x$ is discts at $0$

$\operatorname{sgn} x = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

② $\operatorname{sgn} f$ is cts at $c \neq 0$: $\quad f(x) = \operatorname{sgn} x$, Let $c \neq 0$



Let $c > 0$, $f(c) = 1$

Let $\varepsilon > 0$ Let $\delta = c/2$

$|\operatorname{sgn} x - \operatorname{sgn} c| < \varepsilon$ when $|x - c| < \delta$

$x \in (c-\delta, c+\delta)$

Then $(c-\delta, c+\delta) \subseteq (0, \infty)$

$\therefore \forall x \in (c-\delta, c+\delta)$, $x > 0 \Rightarrow \operatorname{sgn} x = 1$

$\Rightarrow |\operatorname{sgn} x - \operatorname{sgn} c| = |1 - 1| = 0 < \varepsilon$

i.e. $|\operatorname{sgn} x - \operatorname{sgn} c| < \varepsilon$ when $|x - c| < \delta$

$\Rightarrow \operatorname{sgn} x$ is cts at $c > 0$

Similarly, $\operatorname{sgn} x$ is cts at $c < 0$ $\therefore \operatorname{sgn} x$ is cts at $c \neq 0$