21st September 2020

## Continuous functions

(9) DIRICHLET'S FUNCTION

Dirichlet's every where discts fn.

$f: \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Let $c \in \mathbb{R}$    let $c \in \mathbb{Q}$ (||'ly other case $c \in \mathbb{R}-\mathbb{Q}$)

To show: $f$ is discts at $c$

$\forall n, \exists x_n \in \mathbb{R} - \mathbb{Q}$ :

$|x_n - c| < 1/n$

$\Rightarrow x_n \to c$ (by density theorem)

$c - 1/n$, $c$, $c + 1/n$, $x_n$

$x_n \to c$

$f(x_n) \not\to f(c)$, $\downarrow$ $0$ $\neq$

$\forall n, f(x_n) = 0 \because x_n \in \mathbb{R}-\mathbb{Q}$ $\therefore \lim f(x_n) = 0$

But $c \in \mathbb{Q}$ $\therefore f(c) = 1$. Hence $\lim_{n\to\infty} f(x_n) \neq f(c)$

$\therefore \exists x_n : x_n \to c$ but $f(x_n) \not\to f(c)$

$\therefore$ by discont. criterion $f$ is disct at $c \in \mathbb{Q}$

||'ly if $c$ is irrational $f$ is disct at $c$

Hence $f$ is discts on $\mathbb{R}$

$f$ is cts at $c \in A$

$x \to c \Rightarrow f(x) \to f(c)$

seq criterion: $x_n \to c$, $f(x_n) \to f(c)$

$\to$ $1/x$, $x \neq 0$

$\to \operatorname{sgn} x = f(x)$, $c \neq 0$    $\operatorname{sgn} x$ cts at $c$

$\operatorname{sgn} x$ is cts at $c$

$\operatorname{sgn} x \to \operatorname{sgn} c$ as $x \to c$

$\lim x^2 = c^2$    $x^2 \to c^2$ as $x \to c$

$\Rightarrow x^2$ is cts at $c$

$f(x) \to f(c)$ as $x \to c$    $\lim_{x\to c} f(x)$ exist $= f(c)$ exist

$f(x) = \sin 1/x$, $x \neq 0$.    $\boxed{f \text{ is not defined at } 0}$   i.e $f(0)$ d.n.e

Also $\lim_{x\to 0} f(x)$ d.ne. $f$ is discts at $0$

$$F = \begin{cases} \sin 1/x, & x \neq 0 \\ ?, & x = 0 \end{cases}$$

$g(x) = x \sin 1/x$, $x \neq 0$    $\boxed{g \text{ is not def at } 0}$    $g(0)$ d.n.e

However, $\lim_{x\to 0} g(x) = 0$ $\longrightarrow$ $g(0)$

$$G(x) = \begin{cases} g(x), & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$\lim_{x\to 0} G(x) \Rightarrow$

$$\lim_{x\to 0} G(x) = \lim_{x\to 0} x \sin \frac{1}{x} = 0 = G(0) \Rightarrow G \text{ is cts}$$

Remark 5.1.7 & example 5.1.8

If $f^n$ is not defined at $x = c$ & the limit of the $f^n$ at $c$ does not exist then we can not extend $f$ to a $f^n$ which is cts at $c$.

However, if $f$ is not def at $c$ but $\lim_{x\to c} f(x)$ exists then such a $f^n$ can be extended to a $f^n$ which cts $c$ as shown below

5.1.8

eg ① Let $f(x) = \sin \frac{1}{x}$, $x \neq 0$. Then $f(0)$ is not def & $\lim_{x\to 0} f(x)$ d.n.e

$\therefore$ we can not extend $f$ to a $f^n$ $F$ which will be cts at $0$, as for $F$ to be cts at $0$, $\lim_{x\to 0} F(x)$ i.e $\lim_{x\to 0} f(x)$ should exist which is not true.

$$F(x) = \begin{cases} f(x) & x \neq 0 \\ \bigcirc & x = 0 \end{cases}$$

② Let $g(x) = x \sin \frac{1}{x}$, $x \neq 0$. $g(0)$ is not def & $\lim_{x\to 0} g(x) = 0$

Define $G(x) = \begin{cases} g(x) & , x \neq 0 \\ 0, & x = 0 \end{cases}$

Then $\lim_{x\to 0} G(x) = \lim_{x\to 0} g(x) = 0$ & $G(0) = 0$

$\therefore$ $G$ is an extension of $g$ & $G$ is cts at $0$

Ex Q8 $f(x) = \frac{x^2 + x - 6}{x - 2}$, $x \neq 2$. Can $f$ be defined at $x = 2$ in such a way that $f$ is cts at that point

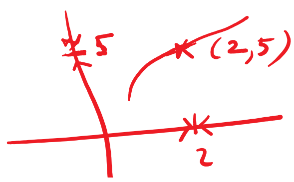
Sol

$$f(x) = \begin{cases} \frac{x^2 + x - 6}{x - 2} & , x \neq 2 \\ 5 & x = 2 \end{cases}$$

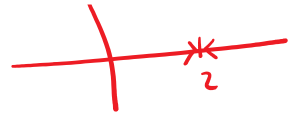
$\lim_{x\to 2} (\ ) = 5$

$(2, 10)$

$\therefore f(2) = 5$ & $\lim_{x\to 2} f(x) = 5$



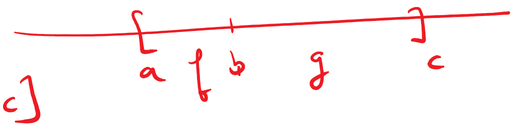
$\therefore f(2) = 5$ & $\lim_{x \to 2} f(x) = 5$



$\therefore \lim_{x \to 2} f(x) = f(2)$

**Q3** Let $a < b < c$. Suppose $f$ is cts on $[a,b]$ & $g$ is cts on $[b,c]$ & $f(b) = g(b)$. Define $h$ on $[a,c]$ as $h(x) = f(x)$, $x \in [a,b]$ & $h(x) = g(x)$, $x \in (b,c]$. Show that $h$ is cts on $[a,c]$


**Sol** Let $p \in [a,c]$




$p \in [a,b]$ or $p \in (b,c]$

Let $p \in [a,b]$. Let $\varepsilon > 0$

Case 1 $p = a$ (or $p = b$)

Let $\varepsilon > 0$

As $f$ is cts at $a$, $\exists\, \delta > 0$:

$|f(x) - f(a)| < \varepsilon$ when $x \in (a, a+\delta)$, $x \in [a,b]$.

$\therefore |h(x) - h(p)| = |f(x) - f(p)|$ $\quad x \in [a,b]$

$= |f(x) - f(a)| < \varepsilon$ when $x \in (a, a+\delta)$

i.e. $|h(x) - h(p)| < \varepsilon$ when $x \in (p, p+\delta)$ & $x \in [a,c]$

$\Rightarrow h$ is cts at $p$.

# $p = b$ is similar

$p \neq a$, $p \neq b$ i.e. $a < p < b$