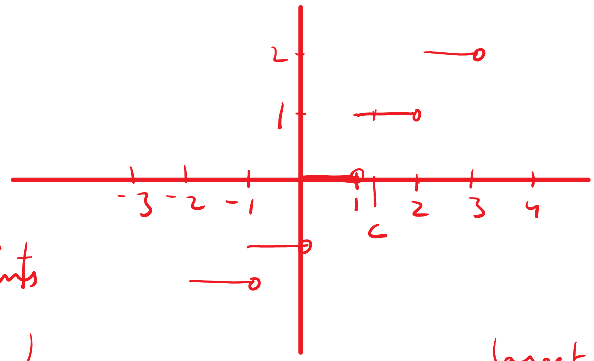
22nd September 2020

Q4 Determine the continuity of

ⓐ $[[x]]$



(We'll show $[[x]]$ is discontinuous at integral points (on $\mathbb{Z}$) and continuous elsewhere)

$f(x) = [[x]]$

$[[x]]$ = the largest integer less than equal to x

$[[1.2]] = 1$ $\quad [[2.9]] = 2$

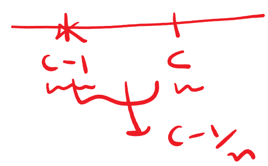
$[[1]] = 1$

(Discont: $x_n \to c$ ⇒ $f(x_n) \not\to f(c)$)

Let $c \in \mathbb{R}$

Let $c \in \mathbb{Z}$ $\quad$ Let $\forall n$, $x_n = c - \frac{1}{n}$

$c - \frac{1}{n} < c$



$\frac{1}{n} \leq 1$

$c - \frac{1}{n} \geq c-1$

Then, $x_n \to c$

Also $f(c) = [[c]] = c$

$f(x_n) = \left[\left[c - \frac{1}{n}\right]\right] = c-1$

$\forall n$, $f(x_n) = c-1$ $\quad \therefore \langle f(x_n) \rangle \to c-1$

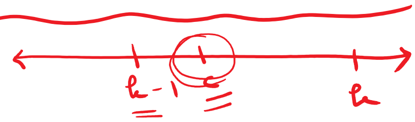
i.e. $\lim f(x_n) = c-1 \neq f(c)$

$\therefore \exists$ a seq $(x_n) \to c$ such that $f(x_n) \not\to f(c)$

$\therefore$ by discont. criterion f is disc. at $c \in \mathbb{Z}$

Case 2

Let $c \notin \mathbb{Z}$



$\exists\, k \in \mathbb{Z} : k-1 < c < k$ $\longrightarrow$ $f(c) = [[c]] = k-1$

Let $x_n \to c$

$f(x_n) = [[x_n]]$

$= k-1 \quad \forall n$ (as $x_n \to c$)

$\therefore \langle f(x_n) \rangle \to k-1 = f(c)$

$\langle f(x_n) \rangle \to f(c)$ whenever $x_n \to c$

$\therefore$ by seq criterion of cont., f is cts at c

ⓑ

$g(x) = x[[x]]$

$g(1) = 1.1$

$g(1.2) = 1.2 \times 1$

$c \in \mathbb{R}$

$\boxed{c \in \mathbb{Z}}$

$x_n \to c$ $\Big\}$ $f(x_n) \not\to f(c)$

$c \in \mathbb{R}$

Case 1 $\boxed{c \in \mathbb{Z}}$

$x_n \to c$ $\quad f(x_n) \not\to f(c)$

$x_n = c - \frac{1}{n} \to c$

$f(x_n) = x_n [[x_n]]$

$= (c - \frac{1}{n})(c-1)$

$f(c) = c^2$

$\to c \cdot (c-1)$

$f(x_n) \to c(c-1) \neq f(c)$ $\therefore f$ is dis cts at $c \in \mathbb{Z}$

Case 2 $\boxed{c \notin \mathbb{Z}}$ f is cts at c

(c) $h(x) = [[\sin x]]$, $x \in \mathbb{R}$

$[[y]]$ $\quad -1, 0 \quad [[-0.5]] = -1$

$y \in \mathbb{Z}$ discont

$y \notin \mathbb{Z}$ cont

h is cts for x such that $\sin x \notin \mathbb{Z}$

h is discts at $x \in \mathbb{R}$: $\sin x \in \mathbb{Z}$

(d) $k(x) = [[\frac{1}{x}]]$, $x \neq 0$ $\quad$ ($\because [[y]]$ is cts $y \notin \mathbb{Z}$, discts $y \in \mathbb{Z}$)

k is cts if $\frac{1}{x} \notin \mathbb{Z}$ $\quad x \neq \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$

k is discts if $\frac{1}{x} \in \mathbb{Z}$

$\frac{1}{x} = 0, \pm 1, \pm 2, \pm 3, \ldots$

or $x = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4} \ldots$

@

Q6 Let $A \subseteq \mathbb{R}$ & let $f: A \to \mathbb{R}$ be cts at $c \in A$. Show that $\forall \varepsilon > 0$ $\exists$ a nbd $V_\delta(c)$ of c such that

if $x, y \in V_\delta(c) \cap A$ then $|f(x) - f(y)| < \varepsilon$

Sol As f is cts at c

$\therefore \exists \delta_1 > 0 : |f(x) - f(c)| < \frac{\varepsilon}{2}$ when $x \in V_{\delta_1}(c) \cap A$ ——— (1)

$\exists \delta_2 > 0 : |f(y) - f(c)| < \frac{\varepsilon}{2}$ when $y \in V_{\delta_2}(c) \cap A$ ——— (2)

Let $\delta = \min(\delta_1, \delta_2)$ Then $\forall x \in V_\delta(c) \cap A \Rightarrow x \in V_{\delta_i}(c) \cap A$, $i = 1, 2$

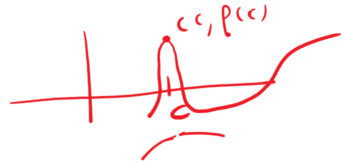
$c - \delta_2 \quad c - \delta_1 \quad c \quad c + \delta_1 \quad c + \delta_2$

$\therefore |f(x) - f(y)| \leq |f(x) - c| + |c - f(y)|$ (Δ ineq)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ (by (1) & (2))

for $x, y \in V_\delta(c) \cap \bar{A}$ $\quad < \varepsilon/2 + \varepsilon/2$ (from ① & ②)

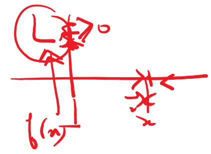
Q7 f is cts at c & $\overset{L}{\uparrow}$ $f(c) > 0$. Show that $\exists$ a nbd $V_\delta(c)$ of c such that $\forall x \in V_\delta(c)$, $f(x) > 0$



Thm 4.2.9

Let $\varepsilon = \frac{f(c)}{2}$. Then $\varepsilon > 0$

$L > 0$



$\exists \delta > 0$ $\quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon \quad \forall x \in V_\delta(c)$

$0 < \frac{f(c)}{2} < f(x)$, $\quad \forall x \in V_\delta(c)$

$\therefore f(x) > 0 \quad \forall x \in V_\delta(c)$