# Ch- 6

## Probability Review

S → denotes sample set

P(A) → denotes probability of A.

Ex!- Suppose the experiment consists of rolling a pair of pair dice. If A is the event that the sum of the dice is equal 7, then

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$.

$P(A) = 6/31 = 1/6$

$P(A^c) = 1 - P(A)$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$P(A|B)$ - Conditional probability of A given that B has occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Ex!- A coin is flipped twice. $S = \{(h,h), (h,t), (t,h), (t,t)\}$ find Conditional probabity that both flips land on heads, given that

(a) the first flip lands on heads

(b) atleast one of the flips lands on heads ?

$\Rightarrow$ $A = \{(h,h)\}$ $\rightarrow$ event that both flips heads.

$B = \{(h,h), (h,t)\}$ $\rightarrow$ first flip heads

$C = \{(h,h), (h,t), (t,h)\}$ $\rightarrow$ atleast one heads.

a $\rightarrow$ $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{2/4} = 1/2$

b $\rightarrow$ $P(A/C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{1/4}{3/4} = 1/3$.

random variables:

Numerical quantities whose values are determined by the outcome of the experiment are known as random variable.

Ex:- Let the random variable $X$ denote the sum when a pair of fair dice are rolled.

possible values of $X = 2, 3, \ldots, 12$.

$P\{X=2\} = P\{(1,1)\} = 1/36$

$P\{X=3\} = 2/36$

$P\{X=4\} = 3/36$

Suppose $x$ is a random variable that can take any one of the finite no. of values say $x_1, x_2, -, x_n$ and $p_i$ represents the probability of occurrence of $x_i$,

then $$\sum_{i=1}^{n} p_i = 1$$

Expected value :- The Expected value of a random variable $x$ is just the average value obtained by regarding the probabilities as frequencies.

$$E(x) = \sum_{i=1}^{m} x_i p_i$$

$E(x)$ is denoted by $\bar{x}$.

$x \rightarrow$ random variable whose possible values are $x_1, x_2, -, x_m$ & $p_i \rightarrow$ probability of $x_i$.

Ex :- roll of dice.

$$\frac{1}{6}(1+2+3+4+5+6) = 3.5.$$

Properties

(i) $E(\alpha y + \beta z) = \alpha E(y) + \beta E(z)$

where $\alpha, \beta$ are real nos. & $y$ & $z$ are random variables.

(ii) $E(x) \geq 0$ if $x \geq 0$.

Variance :- $Var(y) = E[(y-\bar{y})^2]$.
denoted by $\sigma^2$

$$\sigma_y^2 = Var(y)$$

$$\sigma_y = \sqrt{E[(y-\bar{y})^2]}$$

$$Var(x) = E[(x-\bar{x})^2]$$
$$= E[x^2 + \bar{x}^2 - 2x\bar{x}]$$
$$= E(x^2) + E(\bar{x}^2) - 2E(x)\bar{x}$$
$$= E(x^2) - \bar{x}^2$$

Ex:- roll of dice.

$$\bar{y} = 3.5$$

$\sigma^2 = 2.92$. $\sigma = 1.71$.

* Two random variables $x$ & $y$ are said to be independent random variables if the outcome probabilities for one ~~one~~ variable do not depend on the outcome of other.

Ex:- probability of outcome of 4 on second dice is always 1/6 no matter what comes in first dice.

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## Covariance:-

$$\text{Cov}(x_1, x_2) = E[(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)]$$

$$\parallel$$

$$\sigma_{x_1, x_2} \text{ or } \sigma_{12}$$

$$\text{cov}(x_1, x_2) = E(x_1 x_2) - \bar{x}_1 \bar{x}_2$$

* If two random variables $x_1$ & $x_2$ have property that $\sigma_{12} = 0$, then they are said to be uncorrelated.

If $\sigma_{12} > 0$, two variables are said to be positively correlated otherwise negatively correlated.

**Covariance Bound**:- The covariance of two random variables satisfies $|\sigma_{12}| \leq \sigma_1 \sigma_2$

If $\sigma_{12} = \sigma_1 \sigma_2$ → perfectly correlated.

**Correlation coefficient**:- $\rho_{12} = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$

* $|\rho_{12}| \leq 1$

**Variance of sum**

$$\text{var}(x+y) = \sigma_x^2 + 2\sigma_{xy} + \sigma_y^2$$

## Asset Return:-

An investment instrument that can be bought and sold is called an asset.

$$(R)\ \text{Total return} = \frac{\text{amount received } (X_1)}{\text{amount invested } (X_0)}$$

## Rate of return

$$r = \text{rate of return} = \frac{\text{amount received} - \text{amount invested}}{\text{amount invested}}.$$

$$r = \frac{X_1 - X_0}{X_0}$$

$$\Rightarrow \quad R = 1+r$$

or $$X_1 = (1+r)X_0$$

This shows that rate of return acts much like an interest rate.

## Short Sales:-

Short selling or shorting is the sale of a ~~financial~~ asset that is not owned by the seller or that the seller has borrowed.

sell at say $X_0$ & purchase $\rightarrow X_1$

$$\text{Profit} = X_0 - X_1.$$

## Portfolio Return :-

Suppose that $n$ different assets are available and total amount is $X_0$. We than select $X_{0i}$, $i=1,2,\ldots,n$ such that $\sum_{i=1}^{n} X_{0i} = X_0$, where $X_{0i}$ : amount invested in $i^{th}$ ~~Att~~ asset.

[If short selling is allowed $X_{0i}$ become negative).

and $\quad X_{0i} = w_i X_0 \quad i=1,2,\ldots n$

where $w_i$ is the weight or fraction of asset $i$ in the portfolio.

then $\sum_{i=1}^{n} w_i = 1$

(some $w_i$, may be negative if short selling is allowed).

Let $R_i$ denote the total return of asset $i$.

Then amount of money generated by $i^{th}$ asset after end of period $= R_i X_{0i}$

$= R_i w_i X_0$.

$\therefore$ Total amount recieved $= \sum_{i=1}^{n} R_i X_{oi}$

$$= \sum_{i=1}^{n} R_i w_i X_o$$

$\therefore$ total return of portfolio $= \dfrac{\sum_{i=1}^{n} R_i w_i X_o}{X_o}$

$$= \sum_{i=1}^{n} R_i w_i$$

weighted rate

and $r = \dfrac{\sum_{i=1}^{n} R_i w_i X_o - \sum_{i=1}^{n} w_i X_o}{X}$

$$= \frac{\sum_{i=1}^{n} w_i X_o (R_i - 1)}{X}$$

$$= \sum_{i=1}^{n} w_i \frac{(X_o R_i - X_o)}{X_o}$$

$$= \sum_{i=1}^{n} w_i r_i \quad \left( \because \frac{X_o R_i - X_o}{X_o} = r_i \right)$$

Ex:-

| | $r$ | $w_i$ |
|---|---|---|
| A → | 17% | 0.25 |
| B → | 13% | 0.50 |
| C → | 23% | 0.25 |

$$r = \frac{1}{4} \times 17 + \frac{1}{2} \times 13 + \frac{1}{4} \times 23$$

$$= 16.50$$

## Random returns:-

**Ex①** Wheel of fortune $\quad Q_i \to$ payoff by segment $i$.

Expected payoff $= \bar{Q} = \sum_i p_i Q_i = \frac{1}{6}(4-1+2-1+3) = 7/6$

Variance. $\sigma_Q^2 = E(Q^2) - \bar{Q}^2 = \frac{1}{6}(16+1+4+1+9) - (7/6)^2$

$= 3.81$

Total return $= \frac{Q}{1} = Q$

and rate of return, $r = Q - 1$

$\therefore \quad \bar{r} = E(r) = E(Q-1) = E(Q) - 1 = 1/6$

and $\sigma_r^2 = E[(r-\bar{r}^2)] = E[(Q-1-(\bar{Q}^2-1))^2] = \sigma_Q^2 = 3.81.$

**Ex②**: $P(3) = 1/2$, $\quad P(2) = 1/3 \quad P(6) = 1/6$.

white $\leftarrow \bar{R}_1 = \frac{1}{2}(3) + \frac{1}{2}(0) = \frac{3}{2}$

Black $\leftarrow \bar{R}_2 = \frac{1}{3}(2) + \frac{2}{3}(0) = \frac{2}{3}$

Grey $\leftarrow \bar{R}_3 = \frac{1}{6}(6) + \frac{5}{6}(0) = 1$

$\sigma_1^2 = \frac{1}{2}(3^2) - \left(\frac{3}{2}\right)^2 = 2.25$

$\sigma_2^2 = 0.889 = \frac{1}{3}(2)^2 - \left(\frac{2}{3}\right)^2$

$\sigma_3^2 = \frac{1}{6} \times 6^2 - 1 = 5.$

A1 $E(R_1 R_2) = 0$.

$\therefore \sigma_{12} = -\bar{R}_1 \bar{R}_2 = -\frac{3}{2} \times \left(\frac{2}{3}\right) = -1$

$\sigma_{13} = -\frac{3}{2} \times 1 = -1.5$

$\sigma_{23} = -\frac{2}{3}(1) = -0.67$.

## Portfolio Mean & Variance:-

**Mean return of Portfolio :-** Suppose that there are $n$ assets with random rate of returns $r_1, r_2, \ldots, r_n$. and these have expected values $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$.

Suppose we form a portfolio of these $n$ assets using the weights $w_i$, $i = 1, 2, \ldots, n$

then $\quad r = w_1 r_1 + w_2 r_2 + \ldots + w_n r_n$

$$E(r) = E\left(\sum_{i=1}^{n} w_i r_i\right)$$

$$= \sum_{i=1}^{n} E(w_i r_i) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \bar{r}_i.$$

**Variance of portfolio return**

$$\sigma^2 = E\left[(r - \bar{r})^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)\right)\left(\sum_{j=1}^{n} w_j (r_j - \bar{r}_j)\right)\right]$$

$$= E\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)\right]$$

$$= \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \qquad (\sigma_{ii} = \sigma_i^2).$$

Ex:-

$\bar{r}_1 = 0.12$ $\quad \bar{r}_2 = 0.15$

$\sigma_1 = 0.20$ $\quad \sigma_2 = 0.18$ $\quad \sigma_{12} = 0.01$

$w_1 = 0.25$ $\quad w_2 = 0.75$

$$\bar{r} = 0.25 \times 0.12 + 0.75 \times 0.15 = 0.1425$$

and.

$$\sigma^2 = (0.25)^2 \times (0.20)^2 + 0.25 \times 0.75 \times 0.01 + 0.75 \times 0.25 \times 0.01 + (0.75)^2 (0.18)^2 = 0.024475.$$

$$\sigma = 0.1564$$

Diversification :- The process of reducing the variance of the returns of a portfolio by including additional assets in portfolio is called diversification.

Suppose, that there are assets, all of which are mutually uncorrelated. Suppose that the rate of return of each of these assets has mean m and variance $\sigma^2$. Now suppose that a portfolio is constructed by taking equal portions of $n$ of these assets; i.e. $w_i = 1/n$

then

rate of portfolio $\leftarrow$ $r = \sum_{i=1}^{n} w_i r_i = \frac{1}{n} \sum_{i=1}^{n} r_i$

$$\text{then} \quad E(r) = \frac{1}{n} \sum_{i=1}^{n} E(r_i)$$

$$= \frac{1}{n} \times n.m = m.$$

and

$$var(r) = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

$$= \sum_{i=j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} w_i^2 \sigma_{ii}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \frac{1}{n^2} . n\sigma^2 = \frac{\sigma^2}{n}$$

Here expected mean of portifolio is independent of $n$, where variance is dependent on $n$, $\therefore$ variance decreases as $n$ increases.

Now suppose we take correlated assets s.t. $Cov(r_i, r_j) = 0.3\sigma^2$

then

$$Var(r) = E\left[\sum_{i=1}^{n} \frac{1}{n}(r_i - \bar{r})\right]^2$$

$$= \frac{1}{n^2} E\left\{\left[\sum_{i=1}^{n}(r_i - \bar{r})\right]\left[\sum_{j=1}^{n}(r_j - \bar{r})\right]\right\}$$

$$= \frac{1}{n^2}\sum_{i,j=1}^{n}\sigma_{i,j} = \frac{1}{n^2}\left\{\sum_{i=j}\sigma_{ij} + \sum_{i\neq j}\sigma_{ij}\right\}$$

$$= \frac{1}{n^2}\left\{n\sigma^2 + 0.3(n^2-n)\sigma^2\right\}$$

$$= \frac{\sigma^2}{n} + 0.3\sigma^2\left(1-\frac{1}{n}\right)$$

$$= \frac{0.7\sigma^2}{n} + 0.3\sigma^2$$

∴ It is impossible to reduced the variance below $0.3\sigma^2$.

## Mean - Standard Deviation Diagram

Portfolio diagram lemma!. The curve in an $\bar{r}$-$\sigma$ diagram defined by non-negative mixtures of assets 1 and 2 lies within the triangular region defined by two original assets and the point on the vertical axis of height

$$A = \frac{\bar{r}_1\sigma_2 + \bar{r}_2\sigma_1}{\sigma_1 + \sigma_2}.$$

Proof:- The mean rate of return of the portfolio.

$$\bar{r}(\alpha) = (1-\alpha)\bar{r}_1 + \alpha\bar{r}_2.$$

and $$\sigma(\alpha) = \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_{12} + \alpha^2\sigma_2^2} \quad -(1)$$

Using the def$^n$ of correlation coefficient

$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$, eq. (1) become

$$\sigma(\alpha) = \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\rho\sigma_1\sigma_2 + \alpha^2\sigma_2^2}$$

We know that $|\rho| \le 1$, Using $\rho = 1$, we find the upper bound

$$\sigma(\alpha)^* = \sqrt{(1-\alpha)^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2}$$

$$= \sqrt{((1-\alpha)\sigma_1 + \alpha\sigma_2)^2}$$

$$= (1-\alpha)\sigma_1 + \alpha\sigma_2.$$

and using $\rho = -1$, we get lower bound,

$$\sigma(\alpha)_p = \sqrt{(1-\alpha)^2\sigma_1^2 - 2\alpha(1-\alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2}$$

$$= \sqrt{((1-\alpha)\sigma_1 - \alpha\sigma_2)^2}$$

$$= |(1-\alpha)\sigma_1 - \alpha\sigma_2|.$$

Now, the upper bound is linear, ~~and~~ also is mean. Therefore mean & standard deviation move proportionally to $\alpha$ between $\alpha = 0$ & $\alpha = 1$. provided $\rho = 1$.

$\therefore$ The portfolio traces a straight line between 1 & 2.

And for lower bound at $\alpha = \frac{\sigma_1}{\sigma_1 + \sigma_2}$

$$\sigma(\alpha)_p = 0$$

and $\sigma(\alpha)_p = (1-\alpha)\sigma_1 - \alpha\sigma_2$, $\alpha < \frac{\sigma_1}{\sigma_1+\sigma_2}$.

and $\sigma(\alpha)_p = \alpha\sigma_2 - (1-\alpha)\sigma_1$, $\alpha > \frac{\sigma_1}{\sigma_1+\sigma_2}$.

~~this~~ At $\alpha = \frac{\sigma_1}{\sigma_1+\sigma_2}$, $\bar{r} = A = \frac{\bar{r}_1\sigma_2 + \bar{r}_2\sigma_1}{\sigma_1+\sigma_2}$

this gives as line b/w 1 to A & A to 2.

∴ the curve traced out by the portfolio must lie within the shaded region.

## The feasible set

The set of points that correspond to portfolios is called the feasible set or feasible region.

## Minimum - Variance set and Efficient frontier

The left boundary of a feasible set is called minimum - variance set.

There is a special point on this set having minimum variance is called minimum-variance point.

The upper portion of minimum - variance set is termed efficient frontier.

## The Markowitz model

Assume there are $n$ assets. The mean rate of returns are $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$ and the covariances are $\sigma_{ij}$ for $i,j = 1,2,\ldots,n$. A portfolio is defined by a set of $n$ weights $w_i, i=1,2,\ldots,n$ that sum to 1.

We need to find a minimum- variance portfolio. Hence we formulate.

minimize $\frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$

subject to $\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$

$\sum_{i=1}^{n} w_i = 1$.

Solution to this problem

Using Lagrange multipliers $\lambda$ and $\mu$, we form Lagrangian,

$$L = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} - \lambda \left( \sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} \right) - \mu \left( \sum_{i=1}^{n} w_i - 1 \right)$$

then we differentiate $L$ with respect to $w_i$'s and set the derivative to zero.

Consider 2-variable case.

$$L = \frac{1}{2} (w_1^2 \sigma_1^2 + 2 w_1 w_2 \sigma_{12} + w_2^2 \sigma_2^2)$$
$$- \lambda (\bar{r}_1 w_1 + \bar{r}_2 w_2 - \bar{r}) - \mu (w_1 + w_2 - 1)$$

$$\frac{\partial L}{\partial w_1} = \frac{1}{2} [2 w_1 \sigma_1^2 + 2 w_2 \sigma_{12}] - \lambda (\bar{r}_1) - \mu$$

$$\frac{\partial L}{\partial w_2} = \frac{1}{2} [2 w_2 \sigma_2^2 + 2 w_1 \sigma_{12}] - \lambda (\bar{r}_2) - \mu$$

setting the derivatives to zero

$$\sigma_1^2 w_1 + \sigma_{12} w_2 - \lambda \bar{r}_1 - \mu = 0$$

$$\sigma_{21} w_1 + \sigma_2^2 w_2 - \lambda \bar{r}_2 - \mu = 0$$

also, we have $\bar{r}_1 w_1 + \bar{r}_2 w_2 = \bar{r}$

& $w_1 + w_2 = 1$

We solve these eqn for $w_1, w_2, \lambda$ & $\mu$.

~~Examples~~:-

In general, we get the equations.

$$\sum_{j=1}^{n} \sigma_{ij} - \lambda \bar{r}_i - \mu = 0 \quad \text{for } i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1$$

→ Two fund theorem (?)

## Inclusion of a Risk free Asset

Suppose there is a risk free asset with rate of return $r_f$. Consider any other risky asset with rate of return $r$, having mean $\bar{r}$ and variance $\sigma^2$.

then covariance of these two is $E[(r - \bar{r})(r_f - r_f)] = 0$

Suppose these two are combined to form a portfolio using a weight $\alpha$ of risk-free asset and $(1-\alpha)$ of risky asset.

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mean of portfolio = $\alpha r_f + (1-\alpha)\bar{r}$

Variance " = $\alpha^2 \cdot 0 + (1-\alpha)^2\sigma^2$

$\sigma = (1-\alpha)\sigma$

The points representing the portfolio traces out a straight line in the $\bar{r}$-$\sigma$ plane.

The One-fund theorem :- There is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk free asset.

How to find F.

Draw a line between the risk free asset and any point of feasible region and let $\theta$ be the angle b/w the line & horizontal axis then

$$\tan\theta = \frac{\bar{r}_p - r_f}{\sigma_p}$$

To get F, we have to maximize $\theta$ (or $\tan\theta$)

Suppose there n risky assets, with weights $w_1, w_2, \ldots, w_n$ s.t. $\sum_{i=1}^{n} w_i = 1$

$$\bar{r}_p = \sum_{i=1}^{n} w_i \bar{r}_i \quad \text{and} \quad r_f = \sum_{i=1}^{n} w_i r_f$$

$$\tan\theta = \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\left(\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j\right)^{1/2}}$$

$$\frac{\partial \tan\theta}{\partial w_k} = \frac{\left(\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j\right)(\bar{r}_k - r_f) - \sum_{i=1}^{n} w_i(\bar{r}_i - r_f)\frac{\partial}{\partial w_k}\left(\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j\right)}{\left(\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j\right)}$$

Setting the derivative to zero, we get

$$(\bar{r}_k - r_f) = \frac{1}{2}\left(\frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j}\right) \sum_{i=1}^{n} \sigma_{ki} w_i$$

$\rightarrow 2\lambda$

or $$(\bar{r}_k - r_f) = \sum_{i=1}^{n} \sigma_{ki} \lambda w_i$$

Now let $v_i = \lambda w_i$

$$\Rightarrow \sum_{i=1}^{n} \sigma_{ki} v_i = \bar{r}_k - r_f$$

Solve these eqn for $v_i$.

and to get $w_k = \frac{v_k}{\sum v_i}$

Ex:- $\bar{r}_1 = 1$, $\bar{r}_2 = 2$, $\bar{r}_3 = 3$ $r_f = 0.5$ $\sigma^2 = 1$

$v_1 = 1 - 0.5 = 0.5$

$v_2 = 2 - 0.5 = 1.5$

$v_3 = 3 - 0.5 = 2.5$

$w_1 = \frac{1}{9}$ $w_2 = 1/3$ $w_3 = 5/9$ → weight of F

mean = $\frac{1}{9} + \frac{2}{3} + \frac{5}{3} = \frac{1+6+15}{9} = \frac{22}{9}$

Variance =