(6) $\mathbb{Z}_p$ is an Integral Domain, where p is Prime.

Q. List all zero-divisors and units in $\mathbb{Z}_{12}$. Can you see a relationship between the zero-divisors of $\mathbb{Z}_{12}$ and the units of $\mathbb{Z}_{12}$?

Soln: Let A = Set of zero-divisors of $\mathbb{Z}_{12}$

and B = Set of units of $\mathbb{Z}_{12}$

then $A = \{0, 2, 3, 4, 6, 8, 9, 10\}$.

$B = \{1, 5, 7, 11\}$

Here we can see that $A \cap B = \phi$ and $A \cup B = \mathbb{Z}_{12}$

$\therefore$ A and B form a Partition for $\mathbb{Z}_{12}$.

$\Rightarrow$ Zero divisors and units form a Partition for $\mathbb{Z}_{12}$.

Cancellation Property:

Let a, b and c belongs to an integral domain. If $a \neq 0$ and $ab = ac$, then $b = c$.

Proof: $\because ab = ac$

$\Rightarrow a(b-c) = 0$

$\Rightarrow b - c = 0$ [$\because a \neq 0$ & a belongs to an I.D.]

$\Rightarrow b = c.$

Q. Show that a commutative ring with cancellation property has no zero divisors.

Soln. Let commutative ring R has cancellation property.

That is, $ab = ac \Rightarrow b = c$ ——— (1)

TS R has no zero divisors.

Suppose that $a \in R$ is a zero divisor in R

$\therefore a \neq 0$ and there exist $b \in R$, $b \neq 0$ such that $a \cdot b = 0$ ——— (2)

Now $a \cdot b = 0 \Rightarrow b \cdot a = 0 \Rightarrow b \cdot a = 0 = b \cdot 0$

Now $a \cdot b = 0 \Rightarrow b \cdot a = 0 \Rightarrow b \cdot a = 0 = b \cdot 0$

($\because R$ is commutative)

$\Rightarrow b \cdot a = b \cdot 0$

$\Rightarrow a = 0$

(using cancellation property ①)

which contradicts that a is a zero divisor of R.

$\Rightarrow R$ has no zero divisors.

Chapter 12 Questions:

1 to 9, 12 to 20, 22, 27 to 29, 31 to 34, 36 to 42, 45, 46, 49, 50, 51