Chapter-3

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classical equations

1. $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$
the wave equation

2. the heat equation

$$u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0$$

3. the Laplace equation.
$$u_{xx} + u_{yy} + u_{zz} = 0$$

4. Poisson eq^n
$$\nabla^2 u = f(x, y, z)$$

5. the Helmholtz eq^n
$$\nabla^2 u + \lambda u = 0$$

6. The biharmonic eq^n
$$\nabla^4 u = 0 = \nabla^2(\nabla^2 u)$$

7. The bi-harmonic wave eq^n
$$u_{tt} + c^2 \nabla^4 u = 0$$

8. the telegraph eq^n
$$u_{tt} + a u_t + b u = c^2 u_{xx}$$

9. the klein-Gordon- ~~eq~~ eq^n
$$\Box u + \lambda^2 u = 0$$
where

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$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

(10) ~~The Schrodinger eqn~~ ?

Euler's equation for compressible fluid flow.

$$u_t + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p, \quad \rho_t + \operatorname{div}(\rho u) = 0$$

where ρ = fluid density
$p = p(\rho)$

The Vibrating string

Let us consider a streched string of length l fixed at the end points.

We will determine the equation of motion which characterizes the position $u(x,t)$ in the string at any time t after an initial disturbance.

Assumptions

(1) The string is flexible and elastic, i.e. the string cannot resist bending moment and tension

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in the string is always in the direction of tangent to the existing profile of the string

(2) there is no elongation of a single segment of the string,
i.e ~~by no~~ the tension is constant

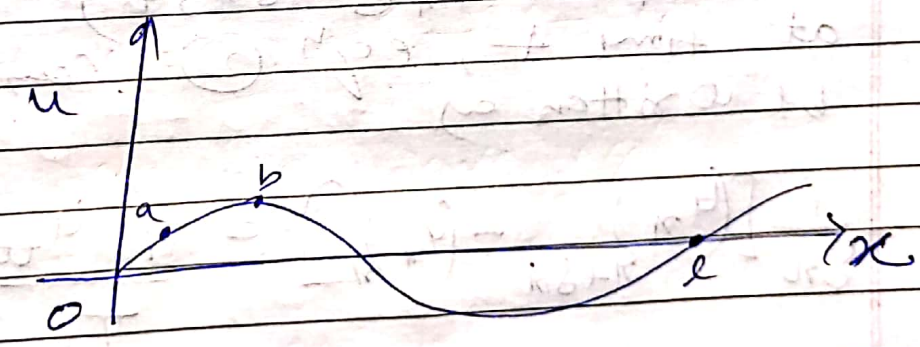
(3) the weight of the string is small compared with the tension in the string

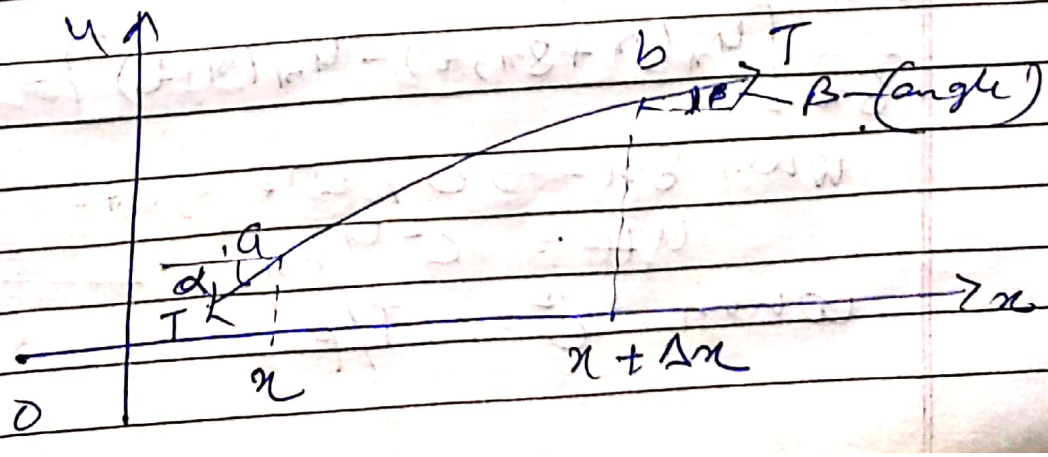
(4) the deflection is small compared with the length of the string

(5) the slope of the displaced string at any point is small

(6) there is only pure transverse Vibration.

Now
let T be the tension at the end point.





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⇒ force on the string in vertical direction is

$$T\sin\beta - T\sin\alpha$$

by Newton's 2nd Law of motion.

$$T\sin\beta - T\sin\alpha = \rho\,\delta s\, u_{tt} \quad \text{—(1)} \qquad F = ma$$

where ρ is the density and δs is the small arc length.

also $\delta s \simeq \delta x$

∵ α & β are small

⇒ $\sin\alpha \simeq \tan\alpha$

$\sin\beta \simeq \tan\beta$

$\alpha \simeq 0$
$\beta \simeq 0$
⇒ $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$
⇒ $\tan\alpha \simeq \sin\alpha$

⇒ from (1)

$$\tan\beta - \tan\alpha = \frac{\rho\,\delta x\, u_{tt}}{T} \quad \text{—(2)}$$

$\tan\theta$ slope at the point x

also $\tan\alpha = u_x(x,t)$

$\tan\beta = u_x(x+\delta x, t)$

at time t, eqn (2) can be written as

$$\frac{1}{\delta x}\left[(u_x)_{x+\delta x} - (u_x)_x\right] = \frac{\rho\, u_{tt}}{T}$$

i.e

$$\frac{1}{\delta x}\left[u_x(x+\delta x, t) - u_x(x,t)\right] = \frac{\rho\, u_{tt}}{T}$$

when $\delta x \to 0$, we get

$$u_{tt} = c^2 u_{xx} \quad \text{—(3)}$$

when $c^2 = T/\rho$

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This is called one-dimensional wave eqn

Now if external force (f) acting on the string then

$$u_{tt} = c^2 u_{xx} + F$$

where $F = f/\rho$

where f may be the pressure, gravitation, resistance etc.

* The Vibrating Membrane :-

Assumptions

1. The membrane is flexible and elastic.
2. There is no elongation of a single segment of the membrane i.e. tension is contant.
3. The weight of the membrane is small compared with the tension in the membrane.
4. The deflection is small compared with the minimal diameter of the membrane.
5. The slope of the displayed membrane at any point is small compared with unity.
6. There is only pure transverse vibration.

We consider a small element of membrane.

∵ deflection & slope are small

so, the area of the element is equal to $\delta x \delta y$.

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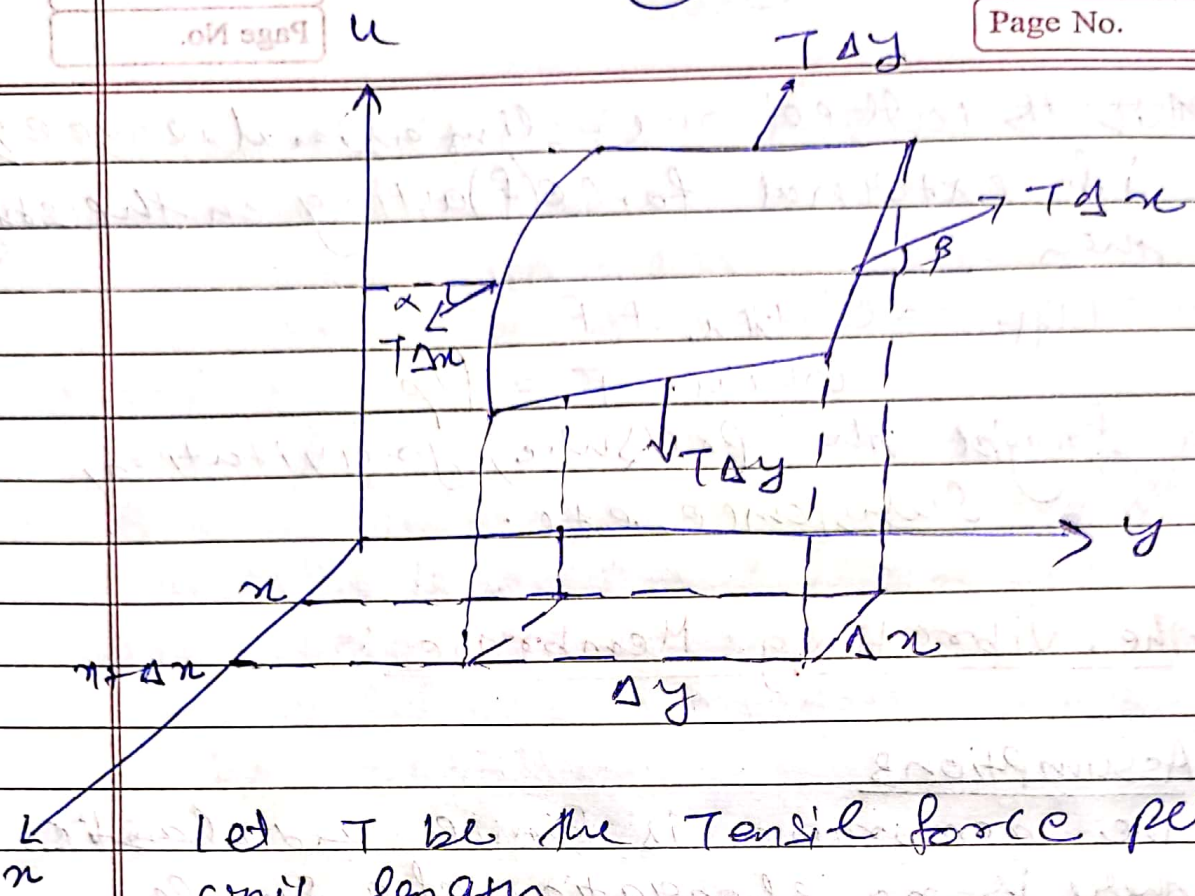

Let T be the Tensil force per unit length

⇒ force acting on the sides of elements are $T\delta x$ and $T\delta y$

⇒ the force acting on the elements of membrane in vertical direction is

$$T\delta x \sin\beta - T\delta x \sin\alpha + T\delta y \sin\delta - T\delta y \sin\gamma$$

∵ slopes are very small

⇒ $\tan\beta = \sin\beta$

$\tan\alpha = \sin\alpha$

⇒ force $= T\delta x(\tan\beta - \tan\alpha) + T\delta y(\tan\delta - \tan\gamma)$

by Newton's 2nd law of motion $\{i.e\ F = ma\}$

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Mass

$$T\delta x(\tan\beta - \tan\alpha) + T\delta y(\tan\delta - \tan\gamma) = \rho\delta A u_{tt} \quad (1)$$

where ρ is the mass per unit area
& $\delta A = \delta x \delta y$.

Now $\tan\alpha = u_y(x_1, y)$
$\tan\beta = u_y(x_2, y+\delta y)$
$\tan\gamma = u_x(x, y_1)$
$\tan\delta = u_x(x+\delta x, y_2)$

where $x_1, x_2 \in (x, x+\delta x)$
$y_1, y_2 \in (y, y+\delta y)$

$\Rightarrow$ put in (1)

$$T\delta x[u_x(x_2, y+\delta y) - u_y(x_1, y)] + T\delta y[u_x(x+\delta x, y_2) - u_x(x, y_1)] = \rho\delta x\delta y u_{tt}$$

$\Rightarrow$

$$\frac{T}{\rho}\left[\frac{u_y(x_2, y+\delta y) - u_y(x_1, y)}{\delta y} + \frac{u_x(x+\delta x, y_2) - u_x(x, y_1)}{\delta x}\right] = u_{tt}$$

when
$\delta x \to 0$, $\delta y \to 0$
we get $u_{tt} = c^2(u_{xx} + u_{yy})$ — (2)
where $c^2 = T/\rho$

This is called two dimensional wave equation.

If there is an external force f per unit area then eq (2) becomes

$$u_{tt} = c^2(u_{xx} + u_{yy}) + F,$$

where $F = f/\rho$

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