(II) $nP_n = xP_n' - P_{n-1}'$

Sol. From generating function

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{---} \; (1)$$

diff (1) w.r.t. z

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$\text{or } (x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} n z^{n-1} P_n(x) \quad \text{---} \; (2)$$

diff (1) w.r.t. x

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2z) = \sum_{n=0}^{\infty} z^n P_n'$$

$$\text{or } z(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n' \quad \text{---} \; (3)$$

(2)/(3)

$$\frac{x-z}{z} = \frac{\sum_{n=0}^{\infty} n z^{n-1} P_n(x)}{\sum_{n=0}^{\infty} z^n P_n'}$$

$$(x-z)\sum_{n=0}^{\infty} z^n P_n' = z\sum_{n=0}^{\infty} n z^{n-1} P_n(x) \quad \text{---} \; (4)$$

equating the coeff. of z^n on both sides of (4)

$$xP_n' - P_{n-1}' = nP_n$$

$$\text{or } \boxed{nP_n = xP_n' - P_{n-1}'} \quad (II)$$

(III) $P'_{n+1} - P'_{n-1} = (2n+1) P_n$

Sol. From relation (I b) we have

$$(n+1) P_{n+1} = (2n+1) x P_n - n P_{n-1} \quad \text{---- (1)}$$

diff. (1) w.r.t. x

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1) x P_n' - n P'_{n-1}$$

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1)\left[n P_n + P'_{n-1} \right] - n P'_{n-1}$$

[from relation (II) $x P_n' = n P_n + P'_{n-1}$]

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1) n P_n + (2n+1) P'_{n-1} - n P'_{n-1}$$

$$(n+1) P'_{n+1} = (2n+1) P_n (1+n) + (2n+1-n) P'_{n-1}$$

collecting the coefficient of P_n & P'_{n-1} we get

$$(n+1) P'_{n+1} = (2n+1)(n+1) P_n + (n+1) P'_{n-1}$$

$$(n+1) P'_{n+1} - (n+1) P'_{n-1} = (n+1)(2n+1) P_n$$

or $\boxed{P'_{n+1} - P'_{n-1} = (2n+1) P_n}$ (III)

of st $n = n-1$ then

$\boxed{P'_n - P'_{n-2} = (2n-1) P_{n-1}}$ (III b)

(IV) $P'_{n+1} - x P'_n = (n+1) P_n$

From relation (Ib) we have

$$(n+1) P_{n+1} = (2n+1) x P_n - n P_{n-1} \quad \text{------ (I)}$$

Diff (I) w.r.to x we get

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1) x P_n' - n P'_{n-1}$$

$$= (2n+1) P_n + (n+1+n) x P_n' - n P'_{n-1}$$
$$= (2n+1) P_n + (n+1) x P_n' + n x P_n' - n P'_{n-1}$$
$$= (2n+1) P_n + (n+1) x P_n' + n (x P_n' - P'_{n-1})$$

$$= (2n+1) P_n + (n+1) x P_n' + n \cdot (n P_n)$$ [since from relation (II) $n P_n = x P_n' - P'_{n-1}$]

$$= (2n+1) P_n + (n+1) x P_n' + n^2 P_n$$
$$= (2n+1+n^2) P_n + (n+1) x P_n'$$
$$= (n+1)^2 P_n + (n+1) x P_n'$$

$$P'_{n+1} = (n+1) P_n + x P_n'$$

or $P'_{n+1} = x P_n' + (n+1) P_n$

or $\boxed{P'_{n+1} - x P_n' = n+1 P_n}$ — (IV)

If st $n = n-1$ we get

$$\boxed{P'_n - x P'_{n-1} = n P_{n-1}} \text{ — (IV b)}$$

2001 ★★★ (V) $(1-x^2)\, P_n' = n(P_{n-1} - x P_n)$

from (IVb)

$$n P_{n-1} = P_n' - x P_{n-1}' \quad \text{---} (1)$$

from (III)

$$n P_n = x P_n' - P_{n-1}' \quad \text{---} (2)$$

multiply (2) by x then subtract from (1) (3)

$$n x P_n = x^2 P_n' - x P_{n-1}' \quad - (3)$$

$$\boxed{n(P_{n-1} - x P_n) = (1-x^2)\, P_n'} \quad \text{---} (V)$$

$n P_{n-1} - n x P_n$
$= P_n' - \cancel{x P_{n-1}'}$
$- x^2 P_n' + \cancel{x P_{n-1}'}$
$\Rightarrow n(P_{n-1} - x P_n)$
$= P_n'(1-x^2)$

$\cancel{n P_n - n x P_{n-1}}$
$\cancel{= x P_n' - P_{n-1}' - x P_n' + x^2 P_{n-1}'}$
$\cancel{\Rightarrow n[P_n - x P_{n-1}]}$
$\cancel{= -[1-x^2] P_{n-1}'}$
$\cancel{(n-1)[P_{n-1} - x P]}$

(VI) $(1-x^2)\, P_n' = (n+1)(x P_n - P_{n+1})$

from (I)

$$(n+1) P_{n+1} = (2n+1) x P_n - n P_{n-1}$$

$$= (n+1+n) x P_n - n P_{n-1}$$

$$\Rightarrow (n+1) P_{n+1} = (n+1) x P_n + n x P_n - n P_{n-1}$$

$$(n+1) P_{n+1} - (n+1) x P_n = n x P_n - n P_{n-1}$$

Or $(n+1)[P_{n+1} - x P_n] = n[x P_n - P_{n-1}]$

$$= -n[P_{n-1} - x P_n] \quad \text{using (V)}$$

$$= -(1-x^2) P_n' \quad \text{(using V)}$$

$$\therefore \boxed{(x^2-1) P_n' = (n+1)[P_{n+1} - x P_n]} \quad \text{---} (VI)$$

or

$$-(1-x^2) P_n' = (n+1)[P_{n+1} - x P_n]$$

$$\Rightarrow (1-x^2) P_n' = (n+1)[x P_n - P_{n+1}].$$

Hence Proved

** Prove that

(i) $P_{2m+1}(0) = 0$ ★

(ii) $P_{2m}(0) = (-1)^m \dfrac{2m!}{2^{2m}(m!)^2}$

Sol (i) $P_n(x) = \dfrac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\left[x^n - \dfrac{n(n-1)}{2\cdot(2n-1)}x^{n-2} + \dfrac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right]$

Set $n = 2m+1$ we get

$$P_{2m+1}(x) = \frac{1\cdot 3\cdot 5 \cdots [2(2m+1)-1]}{(2m+1)!}\left[x^{2m+1} - \frac{(2m+1)(2m+1-1)}{2\cdot[2(2m+1)-1]}x^{2m+1-2} + \cdots\right]$$

Set $x = 0$

we get

$$\boxed{P_{2m+1}(0) = 0}$$

Trick

Note :- If n is odd — $P_n(x)$ contains x

If n is even — $P_n(x)$ free from x

(ii) Sol.

We know that

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{--- (1)}$$

at $x=0$ we have

$$[1+z^2]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(0)$$

or $$\sum_{n=0}^{\infty} z^n P_n(0) = \left[1-(-z^2)\right]^{-1/2}$$

$$= 1+\frac{1}{2}(-z^2) + \frac{1\cdot 3}{2\cdot 4}(-z^2)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(-z^2)^3$$

$$+ \cdots + \frac{1\cdot 3\cdot 5 \cdots (2r-1)}{2\cdot 4\cdot 6 \cdot 2r}(-z^2)^r + \cdots$$

put $n=2m$ as $r=m$

equating the coeff of z^{2m} on either side

$$P_{2m}(0) = \frac{1\cdot 3\cdot 5 \cdots (2m-1)}{2\cdot 4\cdot 6 \cdots 2m}(-1)^m = \frac{1\cdot 3\cdot 5 \cdots (2m-1)(-1)^m}{2^m(1\cdot 2 \cdots m)}$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2m-1)(-1)^m(2\cdot 4\cdot 6 \cdots 2m)}{2^m(m!)\,(2\cdot 4\cdot 6 \cdots 2m)}$$

$$= \frac{2m!\,(-1)^m}{2^m(m!)\,2^m(m!)}$$

$$= \frac{(-1)^m\, 2m!}{2^m\, 2^m m!\, m!}$$

$$\boxed{P_{2m}(0) = (-1)^m \frac{2m!}{2^{2m}(m!)^2}}$$

Bessel's Differential Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

$$\text{or} \quad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

this is called Bessel's equation and particular solution of this eqn are called Bessel's function of order n.

Bessel's function of first kind [$J_n(x)$]

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 + \frac{(-1)\cdot x^2}{2^2\,1!\,(n+1)} + \frac{(-1)^2\, x^4}{2^4\,2!\,(n+1)(n+2)} + \cdots + \frac{(-1)^r\, x^{2r}}{2^{2r}r!\,(n+1)(n+2)\cdots(n+r)}\right]$$

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)} \times \left[\sum_{r=0}^{\infty} \frac{(-1)^r\, x^{2r}}{2^{2r}r!\,(n+1)(n+2)\cdots(n+r)}\right]$$

$$\boxed{J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}}$$

Bessel's function of Second kind

$$\boxed{J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}}$$

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

So General sol. is $\boxed{y = AJ_n(x) + BJ_{-n}(x)}$

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