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Theorem: Let V and W be vector spaces and let $T: V \to W$ be linear. Then T is one-one iff $N(T) = \{0\}$.

Proof: Let T be one-one

Let $x \in N(T) \Rightarrow T(x) = 0$

$\Rightarrow T(x) = T(0)$

$\Rightarrow x = 0$

$\because x \in N(T)$ is an arbitrary element.

$\therefore N(T) = \{0\}$

Conversely $(\Leftarrow)$ Let $N(T) = \{0\}$

TS T is one-one.

Let $T(x) = T(y)$ for some $x, y \in V$

$\Rightarrow \quad T(x) - T(y) = 0$

$\Rightarrow \quad T(x-y) = 0 \qquad [\because T \text{ is linear}]$

$\Rightarrow \quad x - y \in N(T)$

$\Rightarrow \quad x - y = 0 \qquad [\because N(T) = \{0\}]$

$\Rightarrow \quad x = y$

$\therefore$ T is one-one.

~~Theorem:~~ Let V and W be vector spaces of equal (finite) dimension and let $T: V \to W$ be linear. Then the following are equivalent

(a) T is one-to-one

(b) T is onto

(c) $\text{rank}(T) = \dim(V)$

Proof: ~~(a) $\Rightarrow$ (c)~~

T is one-to-one $\Rightarrow$ $N(T) = \{0\}$

$\Rightarrow$ dim. $N(T) = 0$ $\Rightarrow$ nullity $(T) = 0$ —— (1)

from dimension theorem

nullity (T) + rank (T) = dim. (V)

$\Rightarrow$ rank (T) = dim. (V) [using eq (1)]

(c) $\Rightarrow$ (a)

let rank (T) = dim. (V) —— (2)

from dimension theorem.

rank (T) + nullity (T) = dim. (V) [using eq (2)]

$\Rightarrow$ nullity (T) + rank (T) = rank (T)

$\Rightarrow$ nullity $(T) = 0$

$\Rightarrow$ dim. $N(T) = 0$ $\Rightarrow$ $N(T) = \{0\}$.

$\Rightarrow$ T is one-to-one.

(b) $\Rightarrow$ (c) Let T be onto.

$$R(T) = \{T(x) \mid x \in V\} = W.$$

$\Rightarrow$ dim. $R(T) =$ dim. W $\left[\because \text{dim. } V = \text{dim. } W\right]$

$\Rightarrow$ dim. $R(T) =$ dim. V

$\Rightarrow$ rank$(T) =$ dim. V

(c) $\Rightarrow$ (b) Let rank$(T) =$ dim. V

$\Rightarrow$ rank$(T) =$ dim. W

$\Rightarrow$ dim. $R(T) =$ dim. W

$\Rightarrow$ $R(T) = W$ $\left[\because R(T) \subseteq W\right]$

$\Rightarrow$ T is onto.

Thus, all three statements are equivalent

<u>Def^n:</u> let V be a vector space and