(1) # PDE

General example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(2) $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

etc.

Partial derivatives

ex: $u = x^3 + y^3 - 3axy$

$$u(x,y) = x^3 + y^3 - 3axy$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3ax$$

Partial derivatives

Total differentiation

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ du = (3x^2 - 3ay)dx + (3y^2 - 3ax)dy}$$

mostly used in Thermal Physics

PDE

1. Laplace eqn: $\boxed{\nabla^2\phi = 0}$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in Cartesian coordinates

2. Poisson's Eqn $\boxed{\nabla^2\phi = \rho}$

3. Heat flow eqn $\boxed{\nabla^2\phi = \frac{1}{h^2}\frac{\partial\phi}{\partial t}}$

h = constt → called diffusivity

ϕ → non-steady state temp. with no heat source or it may be concentration of a diffusing material

4. Wave Eqn $\boxed{\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}}$

or $\Box^2\phi = 0$ $\qquad$ $\Box$ → called D'Alembertian

where $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\cdot\phi = 0$

5. Helmholtz Diff eqn $\boxed{\nabla^2\phi \pm k^2\phi = 0}$

ϕ → represent the time indep. part of the soln of either the diffusion or wave eqn.

6. Shrodinger Eqn $\boxed{\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0}$ time indep

& $\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$

$\hbar = \frac{h}{2\pi}$, E = total Energy, V = pot en

Sol^n of Laplace eq^n in Cartesian Coordinate

(by method of seperable of variables)

$\partial(u) \to$ dep. vari
$\partial(x) \to$ indep vari

Laplace eq^n is $\nabla^2\phi = 0$, $\nabla^2 u = 0$

ie $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \quad \text{—— (1)}$$

u is function $f(x, y, z)$. using method of seperable of variables we can write

$$u = u(x,y) = X(x)\, Y(y)\, Z(z) \quad \text{—— (2)}$$

ie X is function of x only
Y " " y "
Z " " z "

Now putting (2) in (1) we get

$$YZ\frac{\partial^2 X}{\partial x^2} + ZX\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} = 0 \quad \text{—— (3)}$$

dividing throughout by XYZ

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

or $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} \quad \text{—— (4)}$$

LHS is the function of x only while RHS is the function of y & z only. ∴ each side must be equal to same const k_1^2 (say)

$$\therefore \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k_1^2 \quad \text{or}$$

$$\boxed{\frac{\partial^2 X}{\partial x^2} - k_1^2 X = 0} \quad \text{—— (5)}$$

(4) also $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_1^2 - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$

LHS is functn of y only while RHS is functn of z only

∴ Each side must be = the same constt k_2^2 (say)

$$\therefore \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k_2^2$$

$$\boxed{\frac{\partial^2 Y}{\partial y^2} - k_2^2 Y = 0} \quad -- (6)$$

Similarly $-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - k_1^2 = k_2^2$

$$\text{or } \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -(k_1^2 + k_2^2) = k_3^2 \text{ (say)}$$

$$\text{or } \boxed{\frac{\partial^2 Z}{\partial z^2} - k_3^2 Z = 0} \leftarrow (7)$$

$$\& \ -(k_1^2 + k_2^2) = -(k_1^2 + k_2^2) = k_3^2$$

$$\text{or } \boxed{k_1^2 + k_2^2 + k_3^2 = 0} \ -- (8)$$

∴ the soln of (5), (6) & (7) can be expressed as

$$X = A e^{k_1 x}$$

$$Y = B e^{k_2 y}$$

$$Z = C e^{k_3 z}$$

$$\therefore \boxed{u = XYZ = ABC e^{k_1 x} e^{k_2 y} e^{k_3 z} = ABC e^{k_1 x + k_2 y + k_3 z}} \ (9)$$

Suppose there are infinite no. of k values then the above soln can can be written as

$$\boxed{u = \sum_{k_1, k_2, k_3} N_{k_1 k_2 k_3} e^{k_1 x + k_2 y + k_3 z}}$$

where $N_{k_1 k_2 k_3} = ABC$ arbitrary constt & can be evaluated by using initial conditions

Example

(1) Solve diff-eqn $ax\frac{\partial u}{\partial x} + by\frac{\partial u}{\partial y} = 0$

where a & b are constts.

Soln: Given diff eqn $ax\frac{\partial u}{\partial x} + by\frac{\partial u}{\partial y} = 0$ — (1)

u is fuction of x and y. Using method of separable of variables u can be expressed as

$$u = u(x,y) = X(x)\,Y(y) \ldots \quad \text{— (2)}$$

substituting (2) into (1) we get

$$\Rightarrow ax\frac{\partial}{\partial x}(XY) + by\frac{\partial}{\partial y}(XY) = 0$$

$$\Rightarrow ax\cdot Y\frac{\partial X}{\partial x} + by\,X\frac{\partial Y}{\partial y} = 0$$, Dividing throughout by XY

$$\frac{ax}{X}\frac{\partial X}{\partial x} + \frac{by}{Y}\frac{\partial Y}{\partial y} = 0$$

$$\frac{ax}{X}\frac{\partial X}{\partial x} = +\frac{by}{Y}\frac{\partial Y}{\partial y} \quad \text{— (3)}$$

LHS is fun. of x only while RHS is fun. of y only
If this eqn is satisfied then each side must be equal to same constt, K (say) then

$$\frac{ax}{X}\frac{\partial X}{\partial x} = K \quad \text{— (4)} \quad \& \quad \frac{by}{Y}\frac{\partial Y}{\partial y} = K \quad \text{— (5)}$$

$$\Rightarrow \frac{\partial X}{X} = \frac{K}{a}\cdot\frac{\partial x}{x}$$

Integrating -

$$\log_e X = \frac{K}{a}\log_e x + \log_e C_1$$

or $\log_e X = \log_e (x)^{K/a} + \log_e C_1$

$$\log_e X = \log_e (x)^{K/a}\cdot C_1$$

or $X = C_1 x^{K/a}$ — (6)

similarly ↓

$$Y = C_2\, y^{K/b} \quad \text{— (7)}$$

∴ put (6) & (7) in (1)

$$u(x,y) = C_1 x^{K/a}\cdot C_2 y^{K/b}$$

$$\boxed{u = C\,x^{K/a}\,y^{K/b}}$$ Ans

(6) Ex 2! The diff. eqn is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if $u(x,0) = \sin \pi x$.

Sol: Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ — (1)

Then u depends on t & x, By method of separable of variables we can write

$$u = u(x,t) = X(x)\, T(t) \quad \text{--- (2)}$$

When X is fuction of x only & y " " "y ".

∴ putting (2) in (1) we get

$$\frac{\partial(XT)}{\partial t} = \frac{\partial^2 (XT)}{\partial x^2}$$

$$X \frac{\partial T}{\partial t} = T \frac{\partial^2 X}{\partial x^2}$$ Now divide throughout by XT

$$\frac{1}{T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \quad \text{--- (3)}$$

LHS is fuction of t only while RHS is fuction of X only

∴ each side must be equal to same constt. say $(-k^2)$

∴ $\frac{1}{T}\frac{\partial T}{\partial t} = -k^2$ — (4) and $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2$ — (5)

$$\frac{\partial T}{T} = -k^2 \partial t$$

$$\log_e T = -k^2 t + \log_e C_1$$

$$\log_e T - \log_e C_1 = -k^2 t$$

$$\log_e\left(\frac{T}{C_1}\right) = -k^2 t$$

or $\frac{T}{C_1} = e^{-k^2 t}$

$$\boxed{T = C_1 e^{-k^2 t}} \quad \text{— (6)}$$

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0$$

$$\boxed{X = A\cos kx + B\sin kx} \quad \text{— (7)}$$

$$u(x,t) = C_1 e^{-k^2 t}(A\cos kx + B\sin kx) \quad \text{— (8)}$$

$$u(x,0) = C_1 A\cos kx + C_1 B\sin kx \quad \text{— (9)}$$

Given $u(x,0) = \sin \pi x$ — (10)

∴ $\sin \pi x = C_1 A\cos kx + C_1 B\sin kx$

Compare the coeff. of $\sin kx$ & $\cos kx$ on b.s

$AC_1 = 0$, & $BC_1 = 1$ when $k = \pi$

$$\therefore \boxed{u(x,t) = e^{-k^2 t} \sin \pi x} \quad \text{from (8)}.$$