Partial Differential Equations

- Definition
- One of the classical partial differential equation of mathematical physics is the equation describing the conduction of heat in a solid body (Originated in the 18th century). And a modern one is the space vehicle reentry problem: Analysis of transfer and dissipation of heat generated by the friction with earth’s atmosphere.

For example:

- Consider a straight bar with uniform cross-section and homogeneous material. We wish to develop a model for heat flow through the bar.
- Let $u(x,t)$ be the temperature on a cross section located at x and at time t. We shall follow some basic principles of physics:
- A. The amount of heat per unit time flowing through a unit of cross-sectional area is proportional to $\partial u / \partial x$ with constant of proportionality $k(x)$ called the thermal conductivity of the material.

- B. Heat flow is always from points of higher temperature to points of lower temperature.
- C. The amount of heat necessary to raise the temperature of an object of mass "m" by an amount Δu is a "c(x) m Δu", where c(x) is known as the specific heat capacity of the material.
- Thus to study the amount of heat H(x) flowing from left to right through a surface A of a cross section during the time interval Δt can then be given by the formula:

$$H(x) = -k(x)(area \text{ of A})\Delta t \frac{\partial u}{\partial x}(x,t)$$

Likewise, at the point x + Δx, we have

- Heat flowing from left to right across the plane during an time interval Δt is:

$$H(x+\Delta x) = -k(x+\Delta x)(area \text{ of B})\Delta t \frac{\partial u}{\partial t}(x+\Delta x, t).$$

- If on the interval [x, x+Δx], during time Δt , additional heat sources were generated by, say, chemical reactions, heater, or electric currents, with energy density Q(x,t), then the total change in the heat ΔE is given by the formula:

ΔE = Heat entering A - Heat leaving B + Heat generated .

- And taking into simplification the principle C above, $\Delta E = c(x)\, m\, \Delta u$, where $m = \rho(x)\, \Delta V$. After dividing by $(\Delta x)(\Delta t)$, and taking the limits as Δx , and $\Delta t \to 0$, we get:

$$\frac{\partial}{\partial x}\left[k(x)\frac{\partial u}{\partial x}(x,t) \right] + Q(x,t) = c(x)\rho(x)\frac{\partial u}{\partial t}(x,t)$$

- If we assume k, c, ρ are constants, then the eq. Becomes:

$$\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2} + p(x,t)$$

Boundary and Initial conditions

- Remark on boundary conditions and initial condition on u(x,t).
- We thus obtain the mathematical model for the heat flow in a uniform rod without internal sources (p = 0) with homogeneous boundary conditions and initial temperature distribution f(x), the followlwing Initial Boundary Value Problem:

One Dimensional Heat Equation

$$\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial x^2}(x,t), 0 < x < L, \ t > 0,$$

$$u(0,t) = u(L,t) = 0, \quad t > 0,$$

$$u(x,0) = f(x), \quad 0 < x < L.$$

The method of separation of variables

- Introducing solution of the form
- $u(x,t) = X(x)\,T(t)$.
- Substituting into the I.V.P, we obtain:

$$X(x)T'(t) = \beta\, X''(x)T(t), \quad 0 < x < L, \; t > 0.$$

this leads to the following eq.

$$\frac{T'(t)}{\beta\, T(t)} = \frac{X''(x)}{X(x)} = \text{Constants.} \quad \text{Thus we have}$$

$$T'(t) - \beta\, kT(t) = 0 \;\text{ and }\; X''(x) - kX(x) = 0.$$

Boundary Conditions

- Imply that we are looking for a non-trivial solution X(x), satisfying:

$$X''(x) - kX(x) = 0$$

$$X(0) = X(L) = 0$$

- We shall consider 3 cases:
- $k = 0$, $k > 0$ and $k < 0$.

- Case (i): $k = 0$. In this case we have
- $X(x) = 0$, trivial solution
- Case (ii): $k > 0$. Let $k = \lambda^2$, then the D.E gives $X'' - \lambda^2 X = 0$. The fundamental solution set is: $\{ e^{\lambda x}, e^{-\lambda x} \}$. A general solution is given by: $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$
- $X(0) = 0 \Rightarrow c_1 + c_2 = 0$, and
- $X(L) = 0 \Rightarrow c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0$, hence
- $c_1 (e^{2\lambda L} - 1) = 0 \Rightarrow c_1 = 0$ and so is $c_2 = 0$.
- Again we have trivial solution $X(x) \equiv 0$.

Finally Case (iii) when $k < 0$.

- We again let $k = -\lambda^2$, $\lambda > 0$. The D.E. becomes:
- $X''(x) + \lambda^2 X(x) = 0$, the auxiliary equation is
- $r^2 + \lambda^2 = 0$, or $r = \pm \lambda i$. The general solution:
- $X(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$ or we prefer to write:
- $X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$. Now the boundary conditions $X(0) = X(L) = 0$ imply:
- $c_1 = 0$ and $c_2 \sin \lambda L = 0$, for this to happen, we need $\lambda L = n\pi$, i.e. $\lambda = n\pi/L$ or $k = -(n\pi/L)^2$.
- We set $X_n(x) = a_n \sin(n\pi/L)x$, $n = 1, 2, 3, \ldots$

Finally for $T'(t) - \beta k T(t) = 0, \; k = -\lambda^2$.

- We rewrite it as: $T' + \beta \lambda^2 T = 0$. Or $T' = -\beta \lambda^2 T$. We see the solutions are

$$T_n(t) = b_n e^{-\beta (n\pi/L)^2 t}, \quad n = 1, 2, 3, \ldots$$

Thus the function $u_n(x,t) = X_n(x) T_n(t)$ satisfies the D.E and the boundary conditions. To satisfy the initial condition we try:

$$u(x,t) = \sum u_n(x,t), \text{ over all n.}$$

- More precisely,

$$u(x,t) = \sum_1^\infty c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}\right)x .$$

We must have :

$$u(x,0) = \sum_1^\infty c_n \sin\left(\frac{n\pi}{L}\right)x = f(x) .$$

- This leads to the question of when it is possible to represent f(x) by the so called
- Fourier sine series ??

Example

- Solve the following heat flow problem

  $$\frac{\partial u}{\partial t} = 7\frac{\partial^2 u}{\partial x^2} \; , \quad 0 < x < \pi \; , \quad t > 0.$$

  $$u(0,t) = u(\pi,t) = 0 \, , \quad t > 0 \, ,$$

  $$u(x,0) = 3\sin 2x - 6\sin 5x \; , \; 0 < x < \pi.$$

- Write 3 sin 2x - 6 sin 5x = $\sum c_n \sin (n\pi/L)x$, and comparing the coefficients, we see that $c_2 = 3$, $c_5 = -6$, and $c_n = 0$ for all other n. And we have $u(x,t) = u_2(x,t) + u_5(x,t)$.

Wave Equation

- In the study of vibrating string such as piano wire or guitar string.

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,$$

$$u(0,t) = u(L,t), \quad t > 0,$$

$$u(x,0) = f(x), \quad 0 < x < L,$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 < x < L.$$

Example:

- f(x) = 6 sin 2x + 9 sin 7x - sin 10x , and
- g(x) = 11 sin 9x - 14 sin 15x.
- The solution is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi\alpha}{L} t + b_n \sin \frac{n\pi\alpha}{L} t] \sin \frac{n\pi x}{L}.$$