Inequalities

Result 1. The function $f(x) = \frac{x}{1+x}$, $x \geq 0$, is monotonically increasing.

Pf. Let $y > x \geq 0$. Then $1+y > 1+x \geq 1$ and so

$$\frac{1}{1+y} < \frac{1}{1+x} \Rightarrow 1 - \frac{1}{1+y} > 1 - \frac{1}{1+x}$$

$$\Rightarrow \frac{y}{1+y} > \frac{x}{1+x} \Rightarrow f(y) > f(x).$$

Result 2. For any $x, y \in \mathbb{R}$, we have

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Pf. Let x and y have the same sign; $x \geq 0$, $y \geq 0$ (w/o loss of generality). Therefore,

$$\frac{|x+y|}{1+|x+y|} = \frac{x+y}{1+x+y} = \frac{x}{1+x+y} + \frac{y}{1+x+y}$$

$$\leq \frac{x}{1+x} + \frac{y}{1+y}$$

$$= \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Next let x and y have different signs. We may assume that $|x| \geq |y|$. Then $|x+y| \leq |x|$.

In view of Result 1, $\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$.

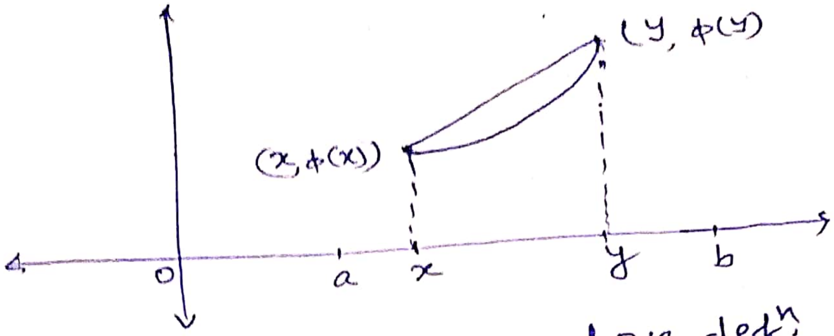
Result 3. "AM- GM Inequality"

If $a > 0$, $b > 0$ and $\lambda \in (0, 1)$ is fixed. Then,

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b.$$

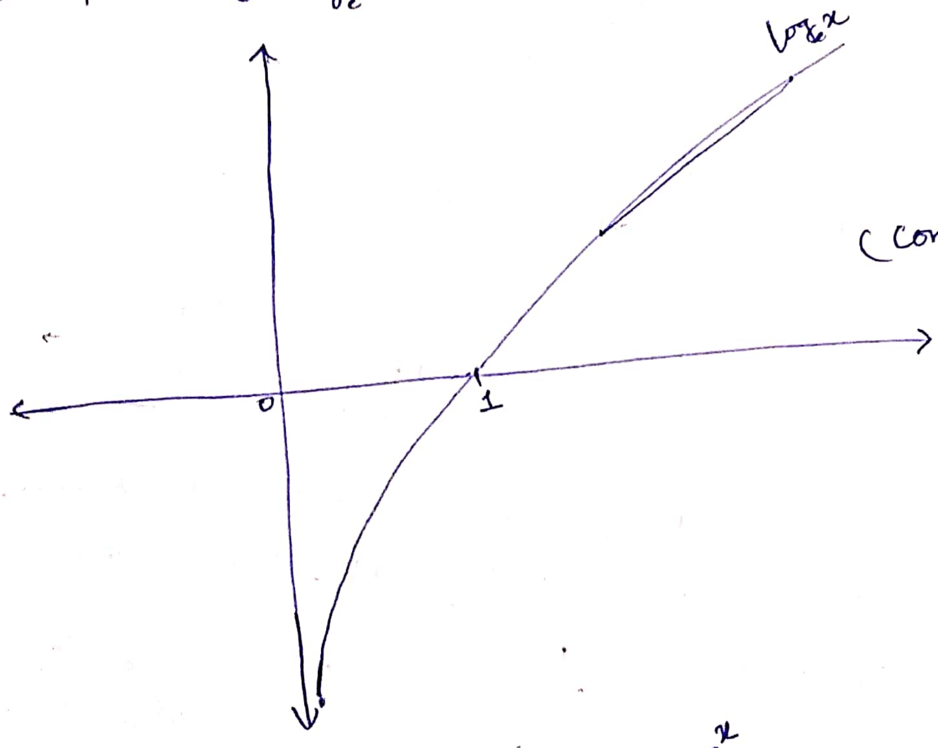
[Convex function: A function $\phi : (a, b) \longrightarrow \mathbb{R}$ is said to be convex if for each $x, y \in (a, b)$ and each $\lambda \in [0, 1]$ we have

$$\phi(\lambda x + (1-\lambda) y) \le \lambda \phi(x) + (1-\lambda)\phi(y)]$$



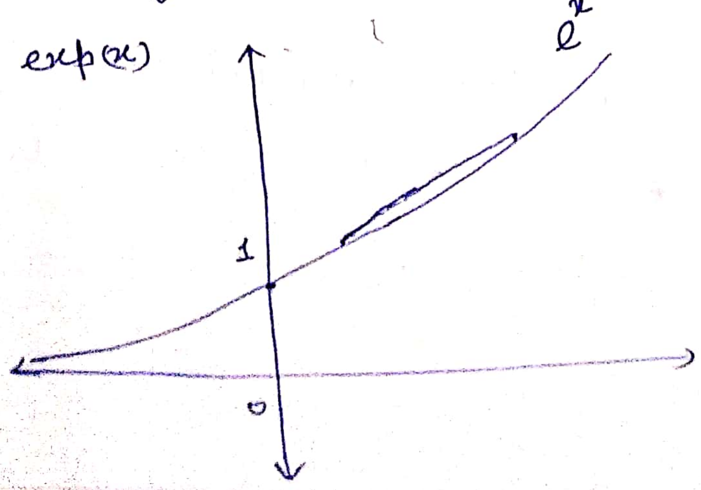
If the inequality in the above defn is reversed the function is said to be concave

example: (i) $y = \log_e x$ $(x > 0)$



(concave)

(ii) $y = \exp(x)$



(Convex)

(Pf. of Result 3)

Since $y = \ln x$ $(x>0)$, is concave, we have

$$\ln(\lambda a + (1-\lambda)b) \geq \lambda \ln a + (1-\lambda)\ln b,$$

i.e.,

$$\ln a^{\lambda} b^{(1-\lambda)} \leq \ln(\lambda a + (1-\lambda)b).$$

As $y = \exp(x)$ is a strictly increasing function, it follows from the above inequality that

$$a^{\lambda} b^{(1-\lambda)} \leq \lambda a + (1-\lambda)b.$$

Alternative proof:

If $a=b$, then the inequality is obvious, in fact, in this case the inequality is actually an equality.

Let $a \neq b$ and $a<b$ (to be specific). Then consider $f(x) = x^{1-\lambda}$ defined on $[a,b]$. Then $\exists$ $c \in (a,b)$ such that

$$\frac{b^{1-\lambda} - a^{1-\lambda}}{b-a} = (1-\lambda)c^{-\lambda} \text{ (In view of}$$

Lagrange's Mean Value Theorem)

$$\Rightarrow b^{1-\lambda} - a^{1-\lambda} = (b-a)(1-\lambda)c^{-\lambda} < (b-a)(1-\lambda)a^{-\lambda}$$

(as $a<c<b \Rightarrow a^{\lambda} < c^{\lambda} < b^{\lambda}$)

On multiplying throughout with $a^{+\lambda}$ (>0) we get

$$a^{\lambda} b^{(1-\lambda)} - a < (b-a)(1-\lambda) = b-a-\lambda(b-a)$$

$$\Rightarrow a^{\lambda} b^{(1-\lambda)} < \lambda a + (1-\lambda)b.$$

Remark: When $x \geq 0$, $y \geq 0$, $p>1$ and $1/p + 1/q = 1$, we have

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

The inequality is obvious if either $x=0$ or $y=0$.

Let $x \neq 0 \neq y$. Then $x>0$ and $y>0$.

Scanned with CamScanner

put $a = x^p$, $b = y^q$, $\lambda = 1/p$ in AM-GM inequality we have

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q \qquad (\text{as } 1-\lambda = 1/q)$$

Result 4 (Hölder's Inequality) Let $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, n$ and suppose that $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}.$$

In particular, when $p = q = 2$, the inequality reduces to

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}.$$

(Cauchy-Schwartz Inequality)

Pf. If either $\sum_{i=1}^{n} x_i^p = 0$ or $\sum_{i=1}^{n} y_i^q = 0$, then inequality is clear. Let $\sum_{i=1}^{n} x_i^p \neq 0 \neq \sum_{i=1}^{n} y_i^q$.

Put $\alpha_i = \dfrac{x_i}{\left(\sum_{i=1}^{n} x_i^p\right)^{1/p}}$ and $\beta_i = \dfrac{y_i}{\left(\sum_{i=1}^{n} y_i^q\right)^{1/q}}$; $1 \leq i \leq n$

Then by a consequence of AM-GM Inequality, we have

$$\alpha_i \beta_i \leq \frac{1}{p}\alpha_i^p + \frac{1}{q}\beta_i^q \qquad (1 \leq i \leq n)$$

$$\Rightarrow \frac{x_i y_i}{\left(\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} y_i^q\right)^{1/q}} \leq \frac{1}{p}\frac{x_i^p}{\sum_{i=1}^{n} x_i^p} + \frac{1}{q}\frac{y_i^q}{\sum_{i=1}^{n} y_i^q}$$

Taking summation over i, we get

$$\frac{\sum_{i=1}^{n} x_i y_i}{\left(\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} y_i^q\right)^{1/q}} \le \frac{1}{p}\,\frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^p} + \frac{1}{q}\,\frac{\sum_{i=1}^{n} y_i^q}{\sum_{i=1}^{n} y_i^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} y_i^q\right)^{1/q}.$$

Result 5 (Minkowski's Inequality)

Let $x_i \ge 0$ and $y_i \ge 0$ $(1 \le i \le n)$ and suppose that $p \ge 1$. Then

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/p} \le \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}.$$

Pf. If $p = 1$, the inequality is self-evident. Let $p > 1$. Suppose $q > 1$ be such that $1/p + 1/q = 1$. $\left(1/q = 1 - 1/p = p-1/p \Rightarrow p = q(p-1)\right)$

We have

$$\sum_{i=1}^{n} (x_i + y_i)^p = \sum_{i=1}^{n} (x_i + y_i)(x_i + y_i)^{p-1}$$

$$= \sum_{i=1}^{n} x_i (x_i + y_i)^{p-1} + \sum_{i=1}^{n} y_i (x_i + y_i)^{p-1}$$

$$\le \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} (x_i + y_i)^{(p-1)q}\right)^{1/q} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}\left(\sum_{i=1}^{n} (x_i + y_i)^{(p-1)q}\right)^{1/q}$$

(In view of Hölder's Inequality)

Thus

$$\sum_{i=1}^{n}(x_i+y_i)^p \le \left[\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}\right]\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1/q}$$

$$\Rightarrow \left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1-1/q} \le \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}$$

(provided $\sum_{i=1}^{n}(x_i+y_i)^p \neq 0$, otherwise inequality is obvious)

$$\Rightarrow \left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1/p} \le \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}.$$

Result 6 (Minkowski's Inequality for Infinite Sums)

Let $p \geq 1$ and $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ be sequences of non-negative terms such that $\sum_{n\in\mathbb{N}} x_n^p$ and $\sum_{n\in\mathbb{N}} y_n^p$ are convergent. Then $\sum_{n\in\mathbb{N}}(x_n+y_n)^p$ is cgt. and

$$\left(\sum_{n=1}^{\infty}(x_n+y_n)^p\right)^{1/p} \le \left(\sum_{n=1}^{\infty} x_n^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p\right)^{1/p}.$$

Pf. For any $m\in\mathbb{N}$, we have from Minkowski's Inequality,

$$\left(\sum_{n=1}^{m}(x_n+y_n)^p\right)^{1/p} \le \left(\sum_{n=1}^{m} x_n^p\right)^{1/p} + \left(\sum_{n=1}^{m} y_n^p\right)^{1/p}$$

$$\le \left(\sum_{n=1}^{\infty} x_n^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p\right)^{1/p}.$$

Thus $\left\{\left(\sum_{n=1}^{m}(x_n+y_n)^p\right)^{1/p}\right\}_{m\in\mathbb{N}}$ is an monotonically increasing sequence of non-negative real numbers bounded above by a fixed real number $\left(\sum_{n=1}^{\infty} x_n^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p\right)^{1/p}$. Hence it will converge to its lub. Therefore,

$$\left(\sum_{n=1}^{\infty}(x_n+y_n)^p\right)^{1/p} \le \left(\sum_{n=1}^{\infty} x_n^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} y_n^p\right)^{1/p}.$$

Result 7: Let $p > 1$. For $a \geq 0$ and $b \geq 0$, we have

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

Pf. If either $a = 0$ or $b = 0$, then inequality is clear. So assume $a > 0$ and $b > 0$. Define

$$\phi : (0, \infty) \longrightarrow (0, \infty) \text{ as}$$

$\phi(x) = x^p$; $p > 1$. Then ϕ is strictly increasing and convex function. Therefore,

$$\phi\left(\tfrac{1}{2}a + \tfrac{1}{2}b\right) \leq \tfrac{1}{2}\phi(a) + \tfrac{1}{2}\phi(b)$$

$$\Rightarrow \left(\frac{a+b}{2}\right)^p \leq \frac{a^p}{2} + \frac{b^p}{2}$$

$$\Rightarrow (a+b)^p \leq 2^{p-1}(a^p + b^p).$$