Complex Analysis

Sec 57. Taylor Series

Theorem Suppose that a function f is analytic throughout a disk $|z-z_0|<R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (|z-z_0|<R_0)$$

where
$$(2) \qquad a_n = \frac{f^n(z_0)}{n!} \qquad (n=0,1,2,3,\ldots)$$

That is, series (1) converges to $f(z)$ when z lies in the stated open disk.

Remark: From (1), we note ~~that~~ the expansion of $f(z)$ into Taylor Series about point z_0. It is the ~~famous~~ familiar Taylor series from Calculus, adapted to functions of a complex variable.

With the agreement that

$$f^0(z_0) = f(z_0) \quad \text{and} \quad 0! = 1$$

Series (1) can be written as

$$(3) \qquad f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \cdots \qquad (|z-z_0|<R_0).$$

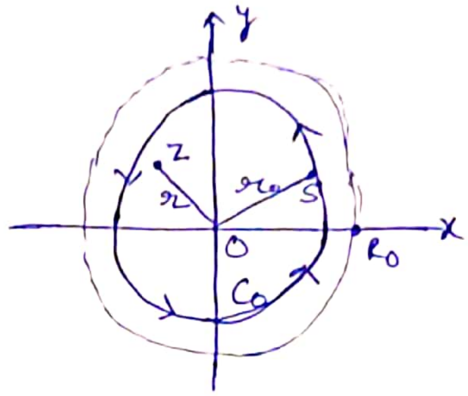
②

Proof. We shall first prove Taylor's Theorem when $z_0 = 0$, in which f is assumed to be analytic throughout the disk $|z| < R_0$

Choose 'z' such that $|z| = r$

Let C_0 denote +vely oriented circle $|z| = r_0$,

where $r < r_0 < R_0$



As f is analytic inside and on the circle C_0.

and Since z is interior to C_0,

By Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s - z} \quad \text{— (i)}$$

We can write

$$\frac{1}{s-z} = \frac{1}{s} \cdot \frac{1}{(1 - z/s)} \quad \text{— (ii)}$$

Now, we know that

$$1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z} \quad (\text{where } z \neq 1)$$

or $$1 + z + z^2 + \cdots + z^N = \frac{1}{1-z} - \frac{z^{N+1}}{1-z}$$

$$\Rightarrow \frac{1}{1-z} = 1 + z + z^2 + \cdots + z^{N-1} + z^N + \frac{z^{N+1}}{1-z}$$

$$= \sum_{n=0}^{N-1} z^n + z^N + \frac{z^{N+1}}{1-z}$$

$$= \sum_{n=0}^{N-1} z^n + \frac{z^N(1-z) + z^{N+1}}{(1-z)}$$

$$= \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}$$

So, we have

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z} \quad (\text{where } z \neq 1)$$

$$\text{— (iii)}$$

Replace z by z/s in (iii), we get

$$\frac{1}{1-z/s} = \sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \frac{(z/s)^N}{1-z/s}$$

$$\Rightarrow \frac{s}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^n} + \frac{z^N s}{(s-z)s^N}$$

$$\Rightarrow \frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + \frac{1}{s^N(s-z)} z^N$$

Multiplying through the above equation by $f(s)$ and then integrating each side with respect to s around C_0, we get

$$\int_{C_0} \frac{f(s)}{s-z}\, ds = \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}} z^n + z^N \int_{C_0} \frac{f(s)\,ds}{(s-z)s^N}$$

Multiply through the above equation by $\frac{1}{2\pi i}$, we get

$$\frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z}\, ds = \sum_{n=0}^{N-1} \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}} z^n + z^N \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{(s-z)s^N}$$

Now using (i) and the fact that $\frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s^{n+1}} = \frac{f^n(0)}{n!}$

$(n = 0, 1, 2, \ldots)$

in the above equation, we get

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{n}(0)}{n!} z^n + \rho_N(z) \quad \text{---- (iv)}$$

where $\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{(s-z)\,s^N}$

We claim that $\lim_{N\to\infty} \rho_N(z) = 0$

Now, ~~As~~ $|z| = r$ and that C_0 has radius r_0 where $r_0 > r$

Then, if s is a point on C_0, we have

$$|s-z| \geq \big||s| - |z|\big| = r_0 - r$$

Now, C_0 is a +ve oriented circle which is closed and bounded set

As f is analytic on C_0

$\therefore$ f is continuous on C_0

$\therefore$ f is bounded on C_0

$\therefore$ $\exists$ a constant $M > 0$ such that

$|f(z)| \leq M$ ~~[illegible]~~ on C_0

$\therefore$ By M-L Theorem,

$$|\rho_N(z)| = \left| \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{(s-z)\,s^N} \right| \leq \frac{r^N}{2\pi} \cdot \frac{M \times 2\pi r_0}{r_0^N\,(r_0 - r)}$$

$$= \left(\frac{r}{r_0}\right)^N \frac{r_0 M}{(r_0 - r)} \to 0$$

as $N \to \infty$

$\left(\because r < r_0 \quad \therefore r/r_0 < 1 \right)$

$\therefore$ From (iv), we get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n \qquad (|z| < R_0) \quad -(v)$$

This is known as <u>Maclaurin series of f</u>.

Now, we will verify the theorem when the disc of radius R_0 is centered at an arbitrary point z_0.

We suppose that f is analytic when $|z - z_0| < R_0$ and note that the composite function $f(z+z_0) = g(z)$ (say) must be analytic when $|(z+z_0) - z_0| < R_0$

i.e. when $|z| < R_0$

Since g is analytic in the disc $|z| < R_0$

$\therefore$ By (v), we have

$$g(z) = \sum_{n=0}^{\infty} \frac{g^n(0)}{n!} z^n \qquad (|z| < R_0)$$

i.e. $$f(z+z_0) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} z^n \qquad (|z| < R_0)$$

Replace 'z' by '$z - z_0$'

Then $$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n \qquad (|z - z_0| < R_0)$$

— (vi)

This is the desired <u>Taylor</u> Series Expansion

Examples

Example 1. Find the Maclaurin series representation of

(a) $f(z) = e^z$ (b) $f(z) = z^2 e^{3z}$

(a) Since $f(z) = e^z$ is entire, it has Maclaurin series expansion valid for all z.

Here $f^n(z) = e^z$ $(n = 0, 1, 2, \ldots)$

$\Rightarrow$ $f^n(0) = 1$ $(n = 0, 1, 2, \ldots)$

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n \qquad \cancel{(|z|<\infty)}$$

$$\Rightarrow \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty) \quad \text{— (1)}$$

Note that if $z = x + i \cdot 0$

Then (1) becomes

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty < x < \infty)$$

(b) Replace 'z' by '$3z$' in (1), and then multiply through the resulting equation by z^2, we get

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} \qquad (|z| < \infty) \quad \text{— (2)}$$

Further, if we replace 'n' by '$n-2$', in (2), we have

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \qquad (|z| < \infty).$$

Example 2. Obtain the Maclaurin series of '$\sin z$'.

Soln: Since the function $f(z) = \sin z$ is entire, it has Maclaurin series representation $\forall z$.

We know that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Moreover, we know that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty) \quad — (1)$$

Replace z by iz in (1), we get

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \quad (|z| < \infty)$$

IIly

$$e^{-iz} = \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \quad (|z| < \infty)$$

$$\therefore \sin z = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] \quad (|z| < \infty)$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty} (1 - (-1)^n) \frac{i^n z^n}{n!}\right] \quad — (2)$$

But $1 - (-1)^n = 0$ when n is even,

so we can replace 'n' by '$2n+1$' in (2)

$$\therefore \sin z = \frac{1}{2i} \sum_{n=0}^{\infty} \left[1 - (-1)^{2n+1}\right] \frac{i^{2n+1} z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

Now $1 - (-1)^{2n+1} = 2$ and $i^{2n+1} = (i^2)^n i = (-1)^n i$

$$\therefore \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty).$$

Illy Maclaurin series of $\cos z$ is given by

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad (|z| < \infty).$$

Example 3. Find the Maclaurin series expansion of $\sinh z$.

Soln. We know that ~~sinh z~~

$$\sinh z = -i \sin(iz)$$

Now, $\sin z = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$

$$\therefore \sin(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n+1}}{(2n+1)!} \quad (|iz| = |z| < \infty)$$

$$\therefore \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty) \quad \text{(verify!)}$$

Illy $\cosh z = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad (|z| < \infty).$

Example 4. Find the Maclaurin series expansion of $\frac{1}{1-z}$ when $|z| < 1$.

Derivatives of the function $f(z) = \frac{1}{1-z}$ which fails to be analytic at $z = 1$ are

$$f^n(z) = \frac{n!}{(1-z)^{n+1}} \quad (n = 0, 1, 2, \ldots)$$

In particular, $f^n(0) = n! \quad (n = 0, 1, 2, \ldots)$

$$\therefore \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} z^n \qquad (|z|<1)$$

$$\Rightarrow \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n$$

or $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

or $$1+z+z^2+\cdots = \frac{1}{1-z} \qquad (|z|<1). \qquad \text{——} (*)$$

Using $(*)$, we can easily see that

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad (|z|<1) \qquad \text{——} (**)$$

Moreover, if we replace variable 'z' in equation $(*)$ by $1-z$, we have the Taylor series rep. of '$\frac{1}{z}$' around the point $z_0=1$

$$\boxed{\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1|<1)}$$

Example 5. Expand the function $f(z) = \dfrac{1+2z^2}{z^3+z^5}$ into series involving powers of z.

Soln.
$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

We can't find Maclaurin series for $f(z)$ since it is not analytic at $z=0$.

We know that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad (|z|<1)$$

$$\therefore \quad \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots \quad (|z|<1)$$

$\therefore$ When $0<|z|<1$

$$f(z) = \frac{1}{z^3}\left(2 - 1 + z^2 - z^4 + z^6 - z^8 + \cdots\right)$$

$$f(z) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - z^5 + \cdots$$

Exercises (Pages 195-197)

Q1. Obtain the Maclaurin series representation

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z|<\infty)$$

Soln.

$$\cosh z = \cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$

$$= \frac{e^{-z}+e^{z}}{2}$$

We know that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z|<\infty)$

$$\therefore \cosh z = \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{(-z)^n}{n!} + \frac{z^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z|<\infty)$$

$$\therefore \cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \qquad (|z|<\infty)$$

Multiply through the above equation by z, we have

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z|<\infty)$$

Q2. Obtain the Taylor Series

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1|<\infty)$$

for the function $f(z)=e^z$ by

(a) using $f^n(1)$ $(n=0,1,2,\dots)$ (b) writing $e^z = e^{z-1}e$.

Soln. a) Since e^z is entire

$\therefore$ it is analytic inside and on any +vely oriented circle around $z_0 = 1$.

$$\therefore \quad e^z = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(z-1)^n \quad \text{where } f(z) = e^z$$

$$(|z-1| < \infty)$$

Now, $f^n(z) = e^z \quad \forall z \in \mathbb{C}, \quad n = 0, 1, 2, \ldots$

$$f^n(1) = e \quad , (n = 0, 1, 2, \ldots)$$

$$\therefore \quad e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

b) We know that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty)$$

Replacing z by $z-1$ in the above equation, We have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

So, $$e^z = e^{z-1} e = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

Q3. Find the Maclaurin series of the function

$$f(z) = \frac{z}{z^4 + 9}$$

Soln.

$$f(z) = \frac{z}{z^4 + 9} = \frac{z}{9} \cdot \frac{1}{(1 + z^4/9)}$$

We know that

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots \qquad (|z| < 1)$$

$$\therefore \frac{1}{1+z^4/9} = \frac{1}{1-(-z^4/9)} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{3^{2n}}$$

when $|z^4/9| < 1$

i.e. $$\frac{1}{1+z^4/9} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{3^{2n}} \quad \text{when} \quad |z| < \sqrt{3}$$

$$\therefore \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+z^4/9} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{3^{2n+2}}$$

when $|z| < \sqrt{3}$.

Q4. Show that if $f(z) = \sin z$, then

$f^{2n}(0) = 0$ and $f^{(2n+1)}(0) = 0$ $(n = 0, 1, 2, \ldots)$

Thus give an alternative derivation of Maclaurin series for $\sin z$.

Soln.

$$f(z) = \sin z$$

$$\left.\begin{aligned} f^{2n}(z) &= (-1)^n \sin z \\ f^{2n+1}(z) &= (-1)^n \cos z \end{aligned}\right\} n = 0, 1, 2, \ldots$$

$$\therefore f^{2n}(0) = (-1)^n . 0 = 0$$

and $f^{2n+1}(0) = (-1)^n$ $\quad n = 0, 1, 2, \ldots$

Since $\sin z$ is an entire function

$$\sin z = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n \quad (|z| < \infty)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad (|z| < \infty).$$

Q.5 Page 196 Do yourself (same as Q4).

Q6 By writing the Maclaurin series for the function $f(z) = \sin(z^2)$

Show that $f^{4n}(0) = 0$ and $f^{(2n+1)}(0) = 0$

$(n = 0, 1, 2, --)$

Soln. The Maclaurin series of $f(z) = \sin z$ is

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty)$$

Replacing z by z^2, we have

$$\sin z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!} \qquad (|z| < \infty)$$

i.e. $$\sin z^2 = \frac{z^2}{1!} - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots \quad \text{— (1)}$$

Since coeff. of z^n in the Maclaurin series of $f(z)$ is $\frac{f^n(0)}{n!}$

i.e. $$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n \qquad (2)$$

Comparing (1) & (2), we observe that

$f^{4n}(0) = 0$

and $f^{2n+1}(0) = 0$ $\quad (n = 0, 1, 2, --)$

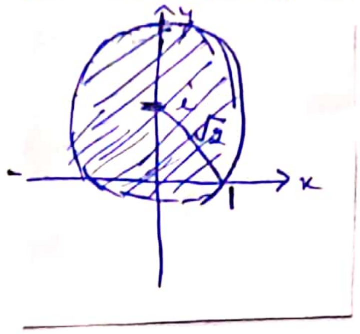
Q7 Derive the Taylor Series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2})$$

Soln. The function $\frac{1}{1-z}$ has a singularity at $z=1$.

$\therefore$ The Taylor series about $z=i$ is valid when $|z-i|<\sqrt{2}$

($\because \frac{1}{1-z}$ is analytic inside and on $|z-i|<\sqrt{2}$)



Consider

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)}$$

$$= \frac{1}{(1-i)}\left[1 - \frac{(z-i)}{(1-i)}\right]$$

(Note that $\left|\frac{z-i}{1-i}\right| < 1$

$\Rightarrow |z-1| < |1-i| = \sqrt{2}$)

$$\frac{1}{1-\left(\frac{z-i}{1-i}\right)} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^n} \quad \text{when} \quad |z-1| < \sqrt{2}$$

$$\therefore \frac{1}{1-z} = \frac{1}{(1-i)} \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^n} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$$

when $|z-i| < \sqrt{2}$

Q8. Using $\cos z = -\sin(z - \pi/2)$

Expand $\cos z$ into Taylor Series about the point $z_0 = \pi/2$.

Soln. We know that $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad |z| < \infty$

$$\therefore \cos z = -\sin(z - \pi/2) = -\sum_{n=0}^{\infty} \frac{(-1)^n (z-\pi/2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (z-\pi/2)^{2n+1}}{(2n+1)!}.$$

Q9. Using the identity $\sinh(z+\pi i) = -\sinh z$ and the fact that $\sinh z$ is periodic with period $2\pi i$, Find the Taylor series for $\sinh z$ about the point $z_0 = \pi i$.

Soln. We know that

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad |z| < \infty \qquad \text{---}(*)$$

It is given that

$$\sinh(z+\pi i) = -\sinh z \quad \text{---}①$$

and $\sinh(z+2\pi i) = \sinh z$ ---② (Period $2\pi i$)

Now,

$$\sinh z = -\sinh(z+\pi i) \qquad (\text{From } ①)$$

$$= -\sinh(z - \pi i + 2\pi i)$$

$$= -\sinh(z - \pi i) \qquad (\text{using } ②)$$

Replacing z by $z - \pi i$ in $(*)$ and then multiply through by -1, we find that

$$\sinh z = -\sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!} \qquad (|z - \pi i| < \infty).$$

~~Q11 Do yourself~~

Q11. Show that when $z \neq 0$

(a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \dfrac{z^2}{4!} + \cdots$

(b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots$

Do yourself

Q12 Derive the expressions

(a) $\dfrac{\sinh z}{z^2} = \dfrac{1}{z} + \sum\limits_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+3)!}$ $\quad (0 < |z| < \infty)$

(b) $z^3 \cosh\left(\frac{1}{z}\right) = \dfrac{z}{2} + z^3 + \sum\limits_{n=1}^{\infty} \dfrac{1}{(2n+2)!} \cdot \dfrac{1}{z^{2n-1}}$

$(0 < |z| < \infty)$

(a) The function $f(z) = \dfrac{\sinh z}{z^2}$ is not analytic at $z=0$.

But we know the expansion

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty)$$

Hence, when $0 < |z| < \infty$

$$f(z) = \frac{1}{z^2}\sinh z = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{z^2}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \right)$$

$$= \frac{1}{z} + \frac{z}{3!} + \frac{z^3}{5!} + \frac{z^5}{7!} + \cdots$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}$$

(b) The function $f(z) = z^3 \cosh\left(\frac{1}{z}\right)$ is not analytic at $z=0$.

But we know the expansion

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

$\therefore$ when $0 < |z| < \infty$

$$\cosh\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{z^{2n}(2n)!}$$

$$\therefore f(z) = z^3 \cosh\left(\frac{1}{z}\right) = z^3 \sum_{n=0}^{\infty} \frac{1}{z^{2n}(2n)!}$$

$$= z^3\left[1 + \frac{1}{z^2 2!} + \frac{1}{z^4 4!} + \cdots\right]$$

$$= z^3 + \frac{z}{2!} + \frac{1}{z.4!} + \frac{1}{z^3.6!} + \cdots$$

$$= \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}}$$

$$(0 < |z| < \infty).$$

Q 13. Show that when $0 < |z| < 4$

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Soln.

$$f(z) = \frac{1}{4z - z^2} = \frac{1}{4z(1 - z/4)}$$

Note that $f(z)$ is not analytic at $z=0$.

But we know the expansion

$$\frac{1}{1 - z/4} = 1 + z/4 + \left(\frac{z}{4}\right)^2 + \cdots \quad \text{when } |z/4| < 1.$$
$$\text{or } |z| < 4.$$

$\therefore$ When $0 < |z| < 4$.

$$f(z) = \frac{1}{4z}\left[1 + z/4 + z^2/16 + \cdots\right] = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$