Group Isomorphism

An mapping $f : G \to G'$ is said to be group isomorphism

if ① f is group homomorphism (i.e preserves the group operation)

② f is onto

③ f is one-one

where G, G' are groups.

Remark : preserves group operation

i.e $f(a*b) = f(a) \cdot f(b) \quad \forall\, a, b \in (G, *)$

$(G, *)$ and $(G', \cdot)$ where $*, \cdot$ are operation of group.

Automorphism :

A mapping $f : G \to G$ is said to be Automorphism

if ① f is homomorphism

② f is one-one

③ f is onto , where G is group.

i.e An isomorphism from a group G onto itself is called automorphism of G.

Example ① $(\mathbb{C}, +)$

consider mapping $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(a+bi) = a-bi$

show : ϕ is an automorphism of group of complex number under addition operation

(Solution) 1-1 & well defined

$\phi(z_1) = \phi(z_2)$ where $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$

$\Leftrightarrow \phi(a_1 + ib_1) = \phi(a_2 + ib_2)$

$\Leftrightarrow \quad a_1 - ib_1 = a_2 - ib_2$ (two complex numbers are equal $\Leftrightarrow$ real & imaginary equal)

$\Leftrightarrow a_1 = a_2$ & $b_1 = b_2$

$\Leftrightarrow a_1 + ib_1 = a_2 + ib_2$

$\Leftrightarrow z_1 = z_2$

Onto: Let $z = a + ib \in \mathbb{C}$

then $\exists z' = a - ib \in \mathbb{C}$

s.t $\phi(a - ib) = a + ib = z$

i.e $\phi(z') = z$

Homomorphism: Let $z_1, z_2 \in \mathbb{C}$

where $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$

consider $\phi(z_1 + z_2) = \phi[(a_1 + ib_1) + (a_2 + ib_2)]$

$= \phi[(a_1 + a_2) + i(b_1 + b_2)]$

$= (a_1 + a_2) - i(b_1 + b_2)$

$= a_1 - ib_1 + a_2 - ib_2$

$= \phi(z_1) + \phi(z_2)$

$\therefore \phi$ is an automorphism under group addition.

Remark $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ where $(\mathbb{C}^*, \cdot)$

$f : \mathbb{C}^* \to \mathbb{C}^*$

$f(a + ib) = a - ib$ is also an automorphism of group of non-zero complex numbers under multiplication.

[i.e $f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$ (homomorphism)

f is 1-1, onto]

Example: $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ group under componentwise addition

$\phi : \mathbb{R}^2 \to \mathbb{R}^2$

as $\phi(a, b) = (b, a)$ is an automorphism.

$\therefore$ 1-1 & well defined

$\phi(a_1,b_1) = \phi(a_2,b_2)$

$\Leftrightarrow (b_1,a_1) = (b_2,a_2)$

$\Leftrightarrow b_1 = b_2,\ a_2 = a_1$

$\Leftrightarrow (a_1,b_1) = (a_2,b_2)$

onto let $(a_2,b_2) \in \mathbb{R}^2$

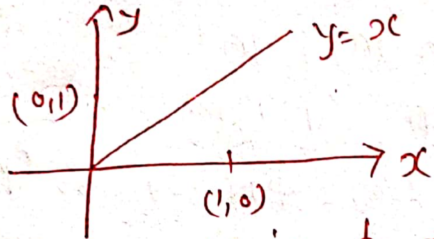
then $\exists\ (b_2,a_2) \in \mathbb{R}^2$

s.t $\phi(b_2,a_2) = (a_2,b_2)$

Homomorphism: $X_1 = (a_1,b_1)$, $X_2 = (a_2,b_2)$

$$\begin{aligned}\phi(X_1+X_2) &= \phi((a_1+a_2),(b_1+b_2)) \\ &= (b_1+b_2,\ a_1+a_2) \\ &= (b_1,a_1)+(b_2,a_2) \\ &= \phi(X_1)+\phi(X_2)\end{aligned}$$

Remark: ① Geometrically, ϕ reflects each point in the plane across the line $y=x$.



Let take point $(1,0)$

then $\phi(1,0) = (0,1)$

$\therefore$ the given mapping ϕ reflects (a,b) through the line $y=x$

$\phi(a,b) = (b,a)$ is reflection ~~across~~ about $y=x$

② Any Reflection across a line passing through the origin or any rotation of plane about the origin is an automorphism of $\mathbb{R}^2$.

Example: Let $G = SL(2,\mathbb{R})$ is group of 2×2 real matrices with determinant 1 with matrix multiplication.

$\phi: SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$

defined as $\phi_M(A) = MAM^{-1}\ \forall A \in G$

and $M \in SL(2,\mathbb{R})$

(i.e M is fixed matrix)

(To show: ϕ is an isomorphism)

(Solution) ϕ is 1-1 & well defined

$\phi_M(A) = \phi_M(B)$

$\Rightarrow MAM^{-1} = MBM^{-1}$

$\Leftrightarrow A = B$ (by pre & post multiply by M)

(Onto): Let $A \in SL(2,\mathbb{R})$

then $\exists\ B \in SL(2,\mathbb{R})$ s.t. $\phi_M(B) = A$

$\Rightarrow MBM^{-1} = A$

$\Rightarrow B = M^{-1}AM$

Also, $|M^{-1}AM| = |M^{-1}||A||M|$

$= 1$

(Homo)

$\phi_M(AB) = M(AB)M^{-1}$

$= MA(M^{-1}M)BM^{-1}$

$= (MAM^{-1})(MBM^{-1})$

$= \phi_M(A)\,\phi_M(B)$ $\therefore \phi_m$ is Automorphism

Remark: ϕ_M is called conjugation by M.

Inner Automorphism

Consider a mapping $\phi: SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$

defined as $\phi_m(A) = MAM^{-1}\ \forall A \in SL(2,\mathbb{R})$

where $M \in SL(2,\mathbb{R})$ is fixed matrix

Let G be any arbitrary group and $a \in G$ (be any fixed elt of G)

and define mapping

$f_a: G \to G$

as $f_a(x) = axa^{-1}\ \forall a \in G$

i.e. for every elt of G, we get a mapping as above, which is also Automorphism.

$\therefore$ This type of mapping is called Inner automorphism.

Inner automorphism induced by a

Let G be a group and let $a \in G$.
The function ϕ_a defined by $\phi_a(x) = a x a^{-1}$ $\forall x \in G$ is called the inner automorphism of G induced by a.

Remark: $\phi_a(x) = a x a^{-1}$ is automorphism of G.
(1-1, onto, Homomorphism)

Example: $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$

D_4 is symmetry of square
D_4 is dihedral group of order 8.

R_0 rotation by angle 0°
R_{90} " " " 90°
R_{180} " " " 180°
R_{270} " " " 270°

H is rotation about Horizontal line
V " " vertical line
D " " about Diagonal

Let $a = R_{90}$
The action of inner automorphism of D_4 induced by R_{90} is given as

$\phi_{R_{90}} : D_4 \to D_4$ defined as $\phi_{R_{90}}(x) = R_{90}\, x\, R_{90}^{-1}$ $\forall x \in D_4$

$x \xrightarrow{\phi_{R_{90}}} R_{90}\, x\, R_{90}^{-1}$

$R_0 \to R_{90} R_0 R_{90}^{-1} = R_{90} R_0 R_{270}$ (270° rotation then 0° then 90° ⇒ total rotation 360°)
$= R_0$

$R_{90} \to R_{90} R_{90} R_{90}^{-1} = R_{90}$

$R_{180} \to R_{90} R_{180} R_{90}^{-1} = R_{180}$

$R_{270} \to R_{90} R_{270} R_{90}^{-1} = R_{270}$

$H \to R_{90} H R_{90}^{-1}$

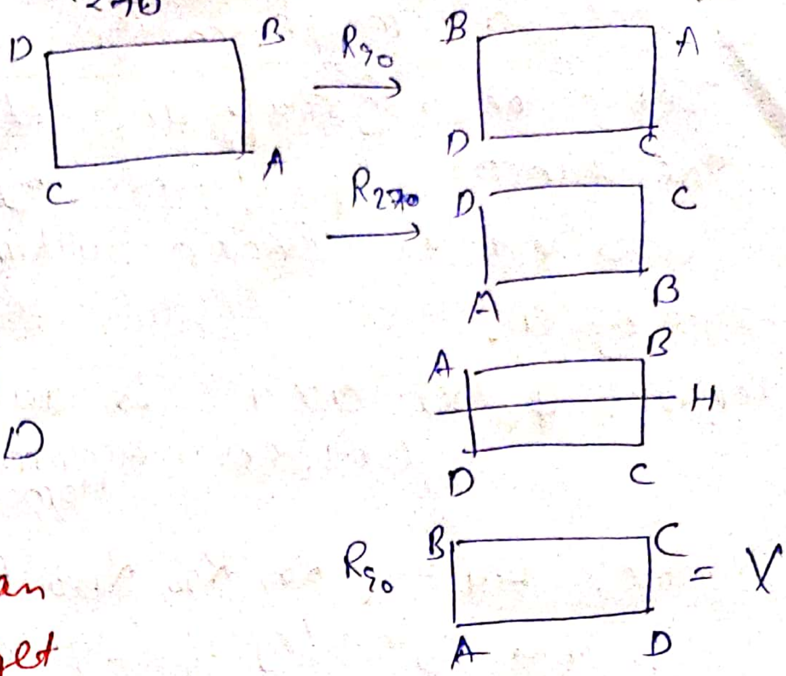
$= V$

$V \to R_{90} V R_{90}^{-1}$

$= H$

$D \to R_{90} D R_{90}^{-1} = D'$

$D' \to R_{90} D' R_{90}^{-1} = D$



Remark: ① Similarly, we can fixed all elts, and get the action of Inner automorphism of D_4.

② Let G be group
Aut(G) is use to denote the set of all automorphism of G
Inn(G) is denote the set of all inner automorphism of G.