Linear equation of the first order :-

The equation of the form

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z) \quad \text{--- (I)}$$

is called linear equation of first order, where P, Q & R are function of x, y, z only. p denotes $\frac{\partial z}{\partial x}$ and q denotes $\frac{\partial z}{\partial y}$.

Example :- (1) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2 + x^2$,

or (2) $(x+y)\frac{\partial z}{\partial x} + (y+z)\frac{\partial z}{\partial y} = z$.

Lagrange's Method :-

The general solⁿ of the linear partial differential equation

$$P(x,y,z)\, p + Q(x,y,z)\, q = R \quad \text{--- (1)}$$

is

$$F(u, v) = 0 \quad \text{--- (2)}$$

where F is an arbitrary function and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ form a solution of the equations

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \quad \text{--- (3)}$$

Example :- Find the general solution of the differential equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y) z \quad \text{--- (1)}$$

By Lagrange's method, we have

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y) z} \quad \text{--- (2)}$$

Taking first two function of eqn (2), we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

By Integration, we obtained

$$-\frac{1}{x} = -\frac{1}{y} - c_1$$

or

$$u(x,y,z) = \boxed{x^{-1} - y^{-1} = c_1} \quad \text{--- (3)}$$

and from eqn (2)

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

or

$$\frac{dx - dy}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\frac{d(x-y)}{x-y} = \frac{dz}{z}$$

By integration, $\log(x-y) = \log(z) + \log c_2$

$$v(x,y,z) = \boxed{\frac{x-y}{z} = c_2} \quad \text{--- (4)}$$

then solⁿ of (1) is

$$f\left(x^{-1} - y^{-1}, \frac{x-y}{z}\right) = 0,$$ where f is an arbitrary function.

Exercise (1) :- Find general soln of following PDE.

$$z(xp - yq) = y^2 - x^2$$

(2) $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

(3) $px(x+y) = qy(x+y) - (x-y)(2x + 2y + z)$

(4) $y^2 p - xyq = x(z - 2y)$

(5) $(y + 2x)p - (x + yz)q = x^2 + y^2$

(6) $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$