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Vector space:

V is a vector space over field F if

1. $(V, +)$ is an abelian group
2. $a \cdot v \in V \quad \forall a \in F, v \in V$ (scalar multiplication)
3. Distributive laws
   $a \cdot (u + v) = a \cdot u + a \cdot v \quad \forall u, v \in V, a \in F$
   $(a+b) \cdot v = a \cdot v + b \cdot v \quad \forall a, b \in F, v \in V.$
4. $a(bv) = (ab)v \quad \forall a, b \in F, v \in V$
5. $\exists\, 1 \in F$ such that $1 \cdot v = v \quad \forall v \in V.$

Examples: $\mathbb{R}^2$ is a vector space over $\mathbb{R}$
$\mathbb{R}^n$ is a vector space over $\mathbb{R}$
$\mathbb{C}$ is a vector space over $\mathbb{R}$.

Subspace test:

$U \neq \emptyset$, $U \subseteq V$ is a subspace of V

if (i) $u - u' \in U \quad \forall u, u' \in U$
(ii) $au \in U \quad \forall a \in F, u \in U$.

Linear Combination:

Let V be a vector space over field F and $v_1, v_2, \dots, v_n$ be any vectors in V.

Any summand of the form

$a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ is called a linear combination of the vectors $v_1, v_2, \dots, v_n$.

Example: In $\mathbb{R}^3$, the vector $(1,2,3)$ is a linear combination of $(0,1,1)$ and $(1,0,1)$

Sol: Let $v_1 = (0,1,1)$, $v_2 = (1,0,1)$

then $v = (1,2,3) = a_1 v_1 + a_2 v_2$.

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$\Rightarrow (1,2,3) = a_1 (0,1,1) + a_2 (1,0,1)$

$\Rightarrow (1,2,3) = (a_2, a_1, a_1 + a_2)$

$$\Rightarrow \left.\begin{aligned} a_2 &= 1 \\ a_1 &= 2 \\ a_1 + a_2 &= 3 \end{aligned}\right\} \Rightarrow a_1 = 2,\ a_2 = 1$$

$\therefore \boxed{v = 2v_1 + v_2} \rightarrow$ Linear combination of v_1 & v_2.

Linear span (or span) :

Let S be a nonempty subset of a vector space V. The span of S is the set consisting of all linear combinations of the vectors in S.

Notation: $\mathrm{Span}(S)$

Note: $\mathrm{Span}(\phi) = \{0\}$.

Suppose $S = \{v_1, v_2, \dots v_n\}$, where $v_i \in V$
$1 \le i \le n$.

then $\text{Span}(S) = \left\{ a_1 v_1 + a_2 v_2 + \dots + a_n v_n \,\middle|\, a_i \in F, 1 \le i \le n \right\}$.

Q. find the span of the set

$S = \{(1,0,0), (0,1,0)\}$ in $\mathbb{R}^3$?

Sol $\text{Span}\, S = \left\{ a_1(1,0,0) + a_2(0,1,0) \,\middle|\, a_1, a_2 \in \mathbb{R} \right\}$

$= \left\{ (a_1, 0, 0) + (0, a_2, 0) \,\middle|\, a_1, a_2 \in \mathbb{R} \right\}$

$= \left\{ (a_1, a_2, 0) \,\middle|\, a_1, a_2 \in \mathbb{R} \right\}$.

= set of all points in xy-plane.

Definition:

A Subset S of a vector space V generates (or spans) V if $\text{Span}(S) = V$.

In this case, we also say that the vectors of S generate V.