

Spherical Wave Equation:-

The wave eqⁿ is given by

$$u_{tt} - c^2 (u_{xx} + u_{yy} + u_{zz}) = 0 \quad \text{--- (1)}$$

eqⁿ (1) in Spherical polar coordinate are
(r, θ , ϕ)

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

Page #115

The wave eqⁿ take the form. Π

$$u_{tt} - c^2 \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right] = 0 \quad \text{--- (2)}$$

if u depends upon r & t only, the solution of this eqⁿ known as spherical symmetrical wave.

$$\Rightarrow u_{tt} - c^2 \left(\frac{1}{r^2} (2r u_r + r^2 u_{rr}) \right) = 0$$

$$\Rightarrow u_{tt} - c^2 \left(\frac{2u_r}{r} + u_{rr} \right) = 0 \quad \text{--- (3)}$$

Let ~~we~~ us assume $U = r u(r, t)$

$$U_t = r u_t$$

$$U_{tt} = r u_{tt}$$

$$\Rightarrow U_r = u + r u_r$$

$$U_{rt} = u_r + U_r + r u_{rt}$$

$$U_{rr} = 2u_r + r u_{rr}$$

so eqⁿ (3) becomes.

$$\frac{1}{r} U_{tt} - c^2 \frac{U_{rr}}{r} = 0$$

$$U_{tt} - c^2 U_{rr} = 0 \quad \text{--- (4)}$$

⇒ Solⁿ of (7)

$$U(r, t) = u = \phi(r+ct) + \psi(r-ct) \quad ; r, ct \geq 0$$

$$\Rightarrow u(r, t) = \frac{1}{2} \psi(r-ct) \quad \rightarrow (4)$$

This Solⁿ of ~~eqn (4)~~ consist of two progressive spherical waves traveling with const. velocity 'c'

Physically the Solⁿ for only outgoing waves generated by a source is of most interest

So the eqn (4) takes the form

$$u(r, t) = \frac{1}{2} \psi(r-ct) \quad - (5)$$

In fluid flow, u represent the velocity potential, u_r is known as radial velocity,

then the total limiting flux through a sphere of centre at the origin & radius r is,

$$q_r = -\frac{1}{r^2} \psi(r-ct) + \frac{1}{r} \psi'(r-ct)$$

$$\text{flux} = Q(t) = \lim_{r \rightarrow 0} 4\pi r^2 u_r$$

$$= \lim_{r \rightarrow 0} 4\pi r^2 \left(\frac{1}{r} \psi'(r-ct) - \frac{1}{r^2} \psi(r-ct) \right)$$

$$= \lim_{r \rightarrow 0} 4\pi (-r \psi'(r-ct) - \psi(r-ct))$$

$$= -4\pi \psi(-ct)$$

$$Q(t) = -4\pi \psi(-ct) \quad - (6)$$

$$\psi(-c(t)) = -\frac{1}{4\pi} Q(t)$$

$$\psi(x-c(t)) = \psi(-c(t-x/c))$$

$$= -\frac{1}{4\pi} Q(t-x/c)$$

$$\Rightarrow u(x,t) = -\frac{1}{c} \frac{1}{4\pi} Q(t-x/c) \quad \text{--- (7)}$$

In fluid flow, the diff. b/w the pressure at any time t and equilibrium is given by

$$p - p_0 = \rho u$$

$$= \frac{-\rho}{4\pi c} Q(t-x/c)$$

where ρ is the density of the fluid

$$\Rightarrow U = x u(x,t) = \frac{1}{2} [(x+t) f(x+t) + (x-t) f(x-t)]$$

$$+ \frac{1}{2c} \int_{x-t}^{x+t} \tau g(\tau) d\tau$$

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x,t) = \phi(x+t) + \psi(x-t)$$

$$u_{tt} - c^2 u_{xx} = 0$$

$$\Rightarrow u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

$$\Rightarrow U = x u(x,0) = x f(x)$$

$$U = x u_t(x,0) = x g(x)$$

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\tau) d\tau$$

for $1 \leq r, r \geq 0$

$$V = \frac{1}{2} [q(t) + \psi(ct - r)]$$

$$0 = \phi(ct) + \psi(ct) \Rightarrow \psi(ct) = -\phi(ct) \quad \text{for } ct > 0$$

$\therefore V_L + \frac{1}{C} V_C$ is const. on each characteristic

$$u_2(x, t) = \frac{1}{2} [\phi'(ct + x) - \phi'(ct - x)]$$

$$U_t = \frac{1}{2} [c \phi'(ct+r) + c \phi'(ct-r)]$$

$$u_2 + \frac{1}{c} u_t = \frac{1}{2} [\phi'(x+r) - \psi'(x-r) + \phi'(x+r) + \psi'(x-r)]$$

now

$$C_2 + \frac{1}{c} U_1 = \frac{1}{2} [(x+ct)f'(x+ct) + f(ct+x) + (x-ct)f'(x-ct) + f(x-ct)] + \frac{1}{c} [cf(x+ct)f'(x+ct)$$

$$+ f(x_2 + \Delta x) - \cancel{f(x_2 - \Delta x)} - \cancel{f'(x_2 - \Delta x) - f'(x_2 - \Delta x)}$$

Integral term.

$$\Rightarrow \phi'(t) = f(t, \phi(t)) + f'(t, \phi(t)) \phi'(t)$$

$$\Rightarrow \phi'(ct) = \cancel{ct f'(ct)} + \cancel{f(ct)} + \frac{c}{c} g(ct)$$

$$= (c-c) f'(ct) + f(ct) + g(ct)$$

5

Date :

Page No.

$$\Rightarrow \phi'(z) = z f'(z) + f(z) + \frac{z}{c} g(z)$$

$$\phi(z) = z f(z) + \int_0^z \frac{\tau}{c} g(\tau) d\tau + \phi(0) K$$

for $t=0$

$$\phi(0) = K$$

$$\Rightarrow \phi(z) = z f(z) + \int_0^z \frac{\tau}{c} g(\tau) d\tau + \phi(0)$$

$$= - \psi(z)$$

$$\phi(t) = t f(t) + \frac{1}{c} \int_0^t \tau g(\tau) d\tau + \phi(0)$$

$$\Rightarrow \psi(t) = -t f(t) - \frac{1}{c} \int_0^t \tau g(\tau) d\tau - \phi(0)$$

$$U(x, t) = \frac{1}{2} [\phi(ct+x) + \psi(ct-x)] = \frac{1}{2} u$$

$$u = \frac{1}{2x} \left\{ [(ct+x) f(ct+x) - (ct-x) f(ct-x)] + \frac{1}{c} \int_{ct-x}^{ct+x} \tau g(\tau) d\tau \right\}$$