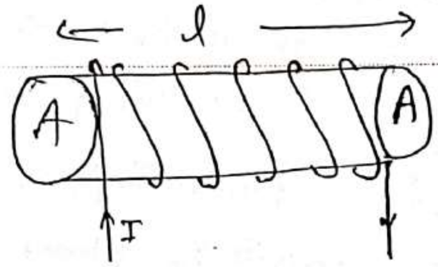
Self Inductance of a long solenoid

Consider a long solenoid of length 'l' meter & a uniform cross-section area 'A' m^2 is placed in air. Let N be the total no. of turns & $n = \frac{N}{L}$ is the no. of turns per unit length, Let a current 'I' is flowing through it, then the mag. induction B inside the solenoid is given by



$$B = \mu_0 n I$$

$$B = \mu_0 \frac{N}{L} I$$

∴ flux through each turn, $\phi_1 = B \cdot A = \frac{\mu_0 N I A}{L}$

∴ Total flux through N turns, $\phi = N \cdot \phi_1 = \frac{\mu_0 N^2 I A}{l}$

∴ self inductance of solenoid, $L = \frac{\phi}{I}$

$$\boxed{L = \frac{\mu_0 N^2 A}{l}}$$

⇒ If the solenoid is wound on a core of constt. permeability μ, then

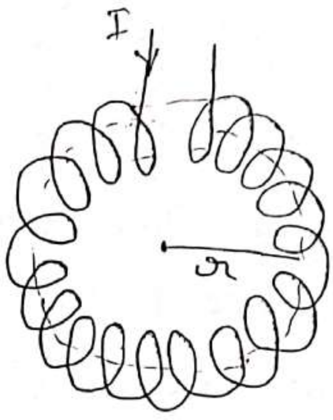
$$L = \frac{\mu N^2 A}{l}, \text{ but } \mu = \mu_0 \mu_r$$

$$\therefore \boxed{L = \frac{\mu_0 \mu_r N^2 A}{l}}$$

⇒ In general if there is a core consisting of a no. of media of relative permeabilities μ_{r_1}, μ_{r_2} --- etc. & area of cross section $A_1, A_2,$ --- etc. then

$$\boxed{L = \frac{\mu_0 N^2}{l}\left[\mu_{r_1} A_1 + \mu_{r_2} A_2 + \mu_{r_3} A_3 + \cdots\right]}$$

Self Inductance of a Toroid



Consider a toroid with uniformly distributed winding of a large no. of closely spaced turns.

Let N be the total no. of turns & l is the length of solenoid & r is radius i.e the mean radius of toroid.

From Ampere's law, the mag. induction B at any point inside the ~~solenoi~~ toroid

$$B = \mu_0 n I$$

$$B = \mu_0 \frac{N}{l} I \qquad \left[\begin{array}{l}\text{Here } \frac{N}{l} = n = \text{no. of turns/length} \\ \& \ l = 2\pi r\end{array}\right]$$

∴ Flux through each turn, $\phi_1 = B \cdot A = \frac{\mu_0 N I A}{l}$

∴ Total flux through N turns, $\phi = N \cdot \phi_1 = \frac{\mu_0 N^2 I A}{l}$

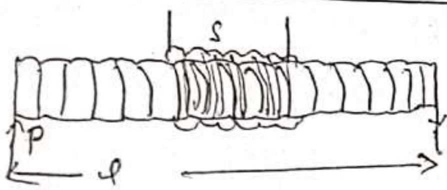
$$\& \quad L = \frac{\phi}{I} = \frac{\mu_0 N^2 A}{l}$$

$$\therefore \boxed{L = \frac{\mu_0 N^2 A}{l}}$$

⇒ If the toroid is wound over a ring of the same area of cross-section & of constt. permeability μ.

then $\boxed{L = \frac{\mu \cdot N^2 A}{l}}$

Mutual Inductance b/w two Co-axial solenoids



consider the secondary coils wound on the central part of primary coil P having an air core.

let N_1 = total no. of turns in the primary coil P, l its length & 'a' its area of cross-section.

N_2 = total no. of turns in the secondary coils.

when a current 'I' flows through the primary then the mag. field on the axis of primary coil P is (or inside)

$$B = \frac{\mu_0 N_1 I}{l}$$

∴ flux through each turn of coil, $\phi_1 = B \cdot A = \frac{\mu_0 N_1 I A}{l}$

since the secondary coil is wound over the central part of the primary, the same flux is also linked with each turn of the secondary.

∴ mag-flux through each turn of secondary also $= \frac{\mu_0 N_1 I A}{l}$

So the Total flux linked with N_2 turns of secondary, $\phi = \frac{\mu_0 N_1 I A}{l} \cdot N_2$

$$\phi = \frac{\mu_0 N_1 N_2 I A}{l}$$

So By definition $\boxed{M = \frac{\phi}{I} = \frac{\mu_0 N_1 N_2 A}{l}}$

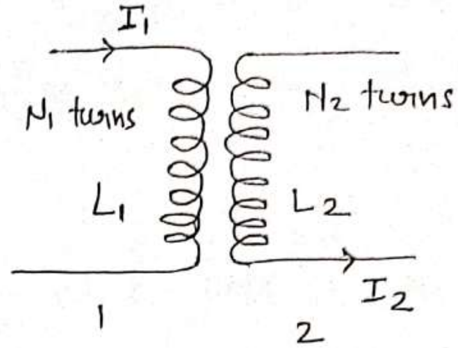
If the primary is closely wound on the core of constt. relative permeability μ_r & that of the same cross section area then

$$\boxed{M = \frac{\mu_0 \mu_r N_1 N_2 A}{l}}$$

If there are a no. of ~~cross~~ cores of area of cross-section A_1, A_2, A_3 etc. & relative permeabilities μ_{r1}, μ_{r2}, μ_{r3} --- etc

$$\boxed{M = \frac{\mu_0 N_1 N_2}{l}\left[\mu_{r1} A_1 + \mu_{r2} A_2 + \mu_{r3} A_3 + \cdots\right]}$$

Coefficient of Coupling



consider two coils 1 & 2 having self inductances L_1 & L_2 and no. of turns N_1 & N_2 also I_1 & I_2 are the current flowing through the two coils 1 & 2 respectively.

Let ϕ_1 & ϕ_2 be the mag. flux linked with each turns of coil 1 & 2, also I_1 & I_2 be the currents flowing in coils 1 & 2 respectively.

∴ Self inductance of coil 1 is $L_1 = \dfrac{N_1 \phi_1}{I_1}$

& self inductance of coil 2 is $L_2 = \dfrac{N_2 \phi_2}{I_2}$ ——— (1)

Suppose ϕ_{21} is the flux linked with each turn of coil 2 when a current I_1 passes through the coil 1 is given by

$$N_2 \phi_{21} = M_{21} I_1$$

$$M_{21} = \frac{N_2 \phi_{21}}{I_1},$$

where M_{21} is the coeff. of mutual inductance of coil 2

similarly, ϕ_{12} is the flux linked with each turn of coil 1 when a current I_2 passes through the coil 2 is given by

$$N_1 \phi_{12} = M_{12} I_2$$

$$M_{12} = \frac{N_1 \phi_{12}}{I_2}$$

where M_{12} is the coeff. of mutual inductance of coil 1

The whole of flux from one coil is linked with the other coil or (the flux due to the field setup by the current in coil 2 is linked with coil 1) or vice versa

then $\phi_{12} = \phi_2$

& $\phi_{21} = \phi_1$

so we get $M = \frac{N_1 \phi_2}{I_2} = \frac{N_2 \phi_1}{I_1}$

[$\because M_{12} = M_{21} = M$ (say)]

$$M^2 = \frac{N_1 N_2 \phi_1 \phi_2}{I_1 I_2} \quad \text{—— (2)}$$

Now $$L_1 L_2 = \frac{N_1 N_2 \phi_1 \phi_2}{I_1 I_2} \quad \text{—— (3)}$$

from (1) & (2) we get $M^2 = L_1 L_2$

$$\boxed{M = \sqrt{L_1 L_2}}$$

Coefficient of Coupling : In above case we have supposed that the entire flux due to either coil links with all the turns of the other i.e. there is no leakage of flux. when this condition is satisfied the coils are said to have a perfect coupling. In general there is some leakage of flux & the coefficient of mutual inductance is $M = K\sqrt{L_1 L_2}$. where K is called coefficient of coupling.

or $$\boxed{K = \frac{M}{\sqrt{L_1 L_2}}}$$

K is a no. b/w 0 & 1, depending upon the geometry of the coil & their relative positions.

When K=1 : the coupling is **perfect** & mutual inductance b/w the coil is maximum. then $M = \sqrt{L_1 L_2}$

When K=0 : there is no coupling b/w two coils. & M=0 is minimum.

Reciprocity Theorem

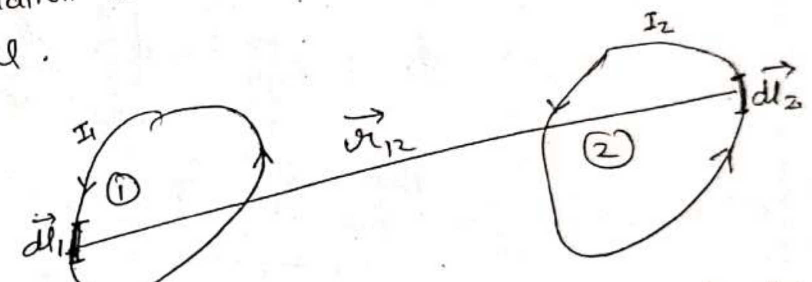
v.Imp

The coefficient of mutual inductance is same whether one or other coil is taken as primary ie $M_{12} = M_{21}$ known as reciprocity theorem.

OR

The mag. flux linked with coil 2 due to any current in coil 1 is exactly equal to the flux linked with coil 1 when the same current is passed through coil 2. ie $M_{12} = M_{21}$

Proof:- To prove this relation we make use of the concept of vector potential.



Consider two coils 1 & 2 and let a current I_1 is pass through coil 1. Let $\vec{B}_1$ denotes the mag. field due to this current in coil 1. Due to current I_1, a mag. flux ϕ_{21} is associated with coil 2. ie

$$\phi_{21} = \int_{S_2} \vec{B}_1 \cdot ds_2 \quad \text{--- (1)}$$

we know that $\vec{\nabla} \cdot \vec{B}_1 = 0$ so $\vec{B}_1$ can be written as

$\vec{B}_1 = \vec{\nabla} \times \vec{A}_1$ where $\vec{A}_1$ is the vector pot. at (ckt) coil 2 due to current I_1 (ie associated with $\vec{B}_1$)

$$\therefore \phi_{21} = \int_{S_2} (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{s}_2 = \oint_2 \vec{A}_1 \cdot \vec{dl_2} \quad \text{--- (2)} \quad \text{(using Stoke's theorem)}$$

$$\phi_{21} = \oint_2 \left(\frac{\mu_0}{4\pi} \oint_1 \frac{I \vec{dl_1}}{r_{12}} \right) \cdot \vec{dl_2} \qquad \text{where } \vec{A}_1 = \frac{\mu_0}{4\pi} \oint_1 \frac{i \vec{dl_1}}{r_1}$$

$$\phi_{21} = \frac{\mu_0}{4\pi} I_1 \oint_2 \oint_1 \frac{\vec{dl_1} \cdot \vec{dl_2}}{r_{12}} \quad \text{--- (3)}$$

where $\vec{r}_{12}$ is the separation from $\vec{dl_1}$ on coil 1 to the point on the coil 2 where potential is A_1 & I_1 donot depends on variables of integration & integration is taken w.r.to stationary ckt.

But $\phi_{21} = M_{21} I_1$ —— (4)

using (3) & (4) we get

$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint_2 \oint_1 \frac{\vec{dl_1} \cdot \vec{dl_2}}{r_{12}}} \quad \text{(A)}$$

This result is called Neumann formula for the coefficient of mutual inductance.

Similarly if a current I_2 is flowing in coil 2 then the mag. flux through coil 1 is given by

$$\phi_{12} = \frac{\mu_0}{4\pi} I_2 \oint_1 \oint_2 \frac{\vec{dl_1} \cdot \vec{dl_2}}{r_{21}}$$

But $\phi_{12} = M_{12} I_2$

$$\therefore \boxed{M_{12} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 \frac{\vec{dl_1} \cdot \vec{dl_2}}{r_{21}}} \quad \text{(B)}$$

Simply $\vec{r}_{12} = \vec{r}_{21} = \vec{r}$ is the seperation b/w $\vec{dl_1}$ & $\vec{dl_2}$

So from (A) & (B) we can write

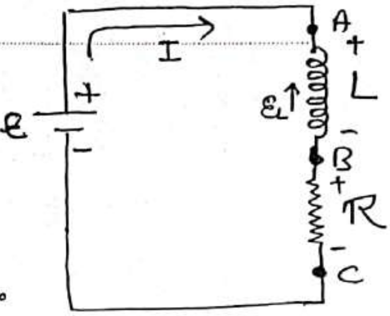
$\boxed{M_{12} = M_{21}}$ → <u>Reciprocity Theorem</u>.

It is clear that

1) mutual induction b/w ~~two~~ two coils depends upon size, shape & relative positions of two coils. Therefore, coeff. of mutual induction change b/w two coils if there is change in size of one or both.
2) Mutual inductance does not depend upon the change in the current in one or both the ckts (coils). Hence change in current in one or both produces no change in coefficent of mutual inductance.
3) Change in the distance b/w two coils changes the mutual inductance.
4) Presence of any non-magnetic material in the neighbourhood of two ckts. (coils) produces no change in mutual inductance.

Energy stored in a Magnetic field

We know that the inductance of a ckt. opposes any change in the current. Suppose in a ckt. current is increasing self inductance will try to decrease the current. Therefore, the work must be done to overcome the induced emf. & drive the current against it. This work done is stored up as the energy in the system.



Consider the ckt.

e = source of emf. connected to resistance R. &

L = inductance

Let $I(t)$ be the current flowing in the ckt. at time t & voltage drop across in the induction is $L\frac{dI}{dt}$ (ie p.d b/w A&B)

By Kirchhoff's law

$$e - L\frac{dI}{dt} - IR = 0 \qquad \left[e - L\frac{dI}{dt} \text{ is net forward emf.}\right]$$

$$e = L\frac{dI}{dt} + IR \quad \text{--- (1)}$$

Suppose a charge dq passes through the source of emf. (battery) in time dt, then the work done by the source in time dt is given by

$$dW = e\,dq$$

$$dW = \left(L\frac{dI}{dt} + IR\right)(I\,dt) \quad \left[\text{Since } dq = I\,dt \text{ \& put } e \text{ from (1)}\right]$$

$$dW = LI\frac{dI}{\cancel{dt}}\cdot \cancel{dt} + I^2R\,dt$$

∴ Total work done in time 't' in which current is changing from 0 to I_{total} is

$$W = \int dW = L\int_0^t I\frac{dI}{dt}\,dt + \int_0^t I^2R\,dt$$

$$W = \frac{1}{2}LI^2 + R\int_0^t I^2\,dt$$

$R\int_0^t I^2\,dt$ → represents the energy dissipated as heat in the resistance

$\frac{1}{2}LI^2$ → energy stored in the inductance.

If energy is not dissipated ie energy donot appear as heat then

$$\boxed{W = \frac{1}{2}LI^2}$$

thus $W = \frac{1}{2} L I^2$ is the energy required from an external source to overcome the emf in the inductance when the current rises to its steady value.

Energy stored up in Solenoid : For solenoid the self inductance is given by $L = \frac{\mu_0 N^2 A}{l}$

& $B = \frac{\mu_0 N I}{l}$

ie $I = \frac{B l}{\mu_0 N}$

But energy stored is given by, $W = \frac{1}{2} L I^2$

$$W = \frac{1}{2}\left(\frac{\mu_0 N^2 A}{l}\right)\left(\frac{B l}{\mu_0 N}\right)^2$$

$$\boxed{W = \frac{1}{2}\frac{B^2 A l}{\mu_0}} \text{ Joules}$$

Energy Density : The energy stored w/in vol $A \cdot l$ is not zero but outside is zero because $B = 0$ outside. Mag. field is uniform so energy is distributed throughout the vol. of solenoid

∴ energy density is the energy per unit vol. of the mag. field

$$\therefore \boxed{\frac{W}{A \cdot l} = \frac{1}{2}\frac{B^2}{\mu_0} \ J/m^2}$$

It is analogous to electrical energy ie energy in electric field which $\frac{\varepsilon_0 \cdot E^2}{2}$