Que :- Obtain the complex form of the fourier series of the function

$$f(x) = \begin{cases} 0 ; & -\pi \le x \le 0 \\ 1 ; & 0 \le x \le \pi \end{cases}$$

Sol :- The complex form of $f(x)$ in fourier series be

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx} \quad \text{--- (1)}$$

where $c_0 = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx\right]$

$$c_0 = \frac{1}{2\pi}\left[x\right]_0^{\pi} = \frac{1}{2} \qquad \therefore \boxed{c_0 = \frac{1}{2}}$$

Now $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \frac{1}{2\pi}\int_{0}^{\pi} 1 \cdot e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$

$$= -\frac{1}{2\pi i n}\left[e^{-in\pi} - 1\right] = \frac{-1}{2\pi i n}\left[\cos n\pi - i\sin n\pi - 1\right]$$

$$= -\frac{1}{2\pi i n}\left[(-1)^n - 0 - 1\right] = \frac{1}{2\pi i n}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 \text{ for } n \text{ is even} \\ \frac{1}{\pi i n} \text{ for } n \text{ is odd} \end{cases} \qquad \& \quad c_{-n} = 0 \text{ for } n \text{ even}, \quad = \frac{1}{\pi i}\left[-\frac{1}{n}\right] \; n = \text{odd}$$

So $f(x) = \frac{1}{2\pi i}\sum_{n=-\infty}^{+\infty} \frac{1-(-1)^n}{n} \cdot e^{inx}$

$$f(x) = \frac{1}{2} + \frac{1}{\pi i}\sum_{n=odd}^{\infty} \frac{1}{n} e^{inx} + \frac{1}{\pi i}\sum_{n=odd} -\frac{1}{n} e^{-inx}$$

$$= \frac{1}{2} + \frac{1}{\pi i}\left[\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots\right] + \frac{1}{\pi i}\left[\frac{e^{-ix}}{-1} + \frac{e^{-i3x}}{-3} + \frac{e^{-5ix}}{-5} + \cdots\right]$$

$$= \frac{1}{2} + \frac{1}{\pi i}\left[(e^{ix} - e^{-ix}) + \frac{1}{3}(e^{3ix} - e^{-3ix}) + \frac{1}{5}(e^{5ix} - e^{-5ix}) + \cdots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}^{\star}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

$\star$ multiply & divide by 2 ie $2\left(\frac{e^{ix} - e^{-ix}}{2i}\right) = 2\sin x$

Parseval's Identity

Assuming that the fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$ then

Parseval's Identity is

$$\frac{1}{l}\int_{-l}^{+l}\{f(x)\}^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

or

$$\boxed{\int_{-l}^{+l}\{f(x)\}^2 dx = l\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]} \quad \text{--- (1)}$$

where the integral is assumed to exist.

we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--- (2)}$$

multiplying (2) by $f(x)$ & integrating w.r.t x b/w $-l$ to l

$$\int_{-l}^{+l}\{f(x)\}^2 dx = a_0\underbrace{\int_{-l}^{+l} f(x)\,dx}_{=2la_0} + \sum_{n=1}^{\infty}\left\{a_n \underbrace{\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}dx}_{=a_n l} + b_n\underbrace{\int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l}dx}_{=b_n l}\right\}$$

as done in change of interval.

$$= a_0[2la_0] + \sum_{n=1}^{\infty}\left[a_n(a_n l) + b_n(b_n l)\right]$$

$$\int_{-l}^{+l}(f(x))^2 dx = \left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right] l$$

or

$$\frac{1}{2l}\int_{-l}^{+l}\{f(x)\}^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

⇒ It is the relation b/w the average of square of the function $f(x)$ & the coefficients in the fourier series of $f(x)$

$$a_0 = \frac{1}{2l}\int_{-l}^{+l} f(x)\,dx \Rightarrow \int_{-l}^{+l} f(x)\,dx = 2la_0$$

$$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l} \Rightarrow \int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}dx = a_n l$$

$$b_n = \frac{1}{l}\int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l} \Rightarrow \int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l}dx = b_n l$$

We know that Parseval's formula is

$$\int_{-l}^{+l} \{f(x)\}^2 dx = l\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

Note if $0 < x < 2l$; then $\int_0^{2l} \{f(x)\}^2 dx = l\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$

(cosine series) $0 < x < l$: $\int_0^{l} \{f(x)\}^2 dx = \frac{l}{2}\left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2\right]$

(sine series) $0 < x < l$: $\int_0^{l} \{f(x)\}^2 dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} b_n^2\right]$

Q:- Expand $f(x) = x^2$, $-\pi < x < \pi$ in fourier series & show that

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Sol:- Let the fourier series of $f(x) = x^2$ be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(x^2 \frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n}dx\right] = \frac{2}{\pi}\left[0 - 0 - \frac{2}{n}\int_0^{\pi} x \sin nx\,dx\right]$$

$$= -\frac{4}{\pi n}\left[\left(-x\frac{\cos nx}{n}\right)_0^{\pi} + \int_0^{\pi} 1\cdot\frac{\cos nx}{n}dx\right] = -\frac{4}{\pi n}\left[\frac{-\pi\cos n\pi}{n} + \left(\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= +\frac{4\cancel{\pi}}{\cancel{\pi} n^2}\cos n\pi = +\frac{4}{n^2}(-1)^n$$

& $b_n = 0$, since function is odd

(i) $\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$ put $x = \pi$

$x^2 =$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

(ii) Applying Parseval's identity

$$\int_{-\pi}^{\pi} \{f(x)\}^2 dx = \pi \left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \right]$$

$$\int_{-\pi}^{\pi} x^4 dx = 2\pi \left(\frac{\pi^2}{3}\right)^2 + \pi \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2}\right]^2$$

$$\frac{2}{5}\pi^5 = \frac{2}{9}\pi^5 + \pi \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{16}{n^4} = \left(\frac{2}{5}\pi^5 - \frac{2}{9}\pi^5\right) \times \frac{1}{\pi} = \frac{8\pi^4}{45}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$ Ans

Q:- Expand $f(x) = x,\ 0 < x < 2$ in half range

(a) sine series & ~~deduce that~~ $\cancel{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots = \frac{\pi^4}{90}}$

(b) cosine series & deduce $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots = \frac{\pi^4}{96}$ using Parseval's identity

Ans. (a) $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$

Here $\ell = 2$

& $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx = \int_0^2 x\sin\frac{n\pi x}{\ell}\,dx$

$$= -x\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\Bigg]_0^2 + \int_0^2 1\cdot\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\,dx$$

$$= -\frac{2}{n\pi}\cdot\left(2\cos n\pi - 0\cdot\cos 0\right) + \left(\frac{2}{n\pi}\right)^2\underbrace{\left(\sin\frac{n\pi x}{2}\right)_0^2}_{=0}$$

$$= -\frac{2}{n\pi}\left[2(-1)^n - 0\right]$$

$$= \begin{cases} -\frac{4}{n\pi}(-1)^n \end{cases}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2} = +\frac{4}{\pi}\sum_{n=1}^{\infty} -\frac{(-1)^n}{n}\sin\frac{n\pi}{2}x$$

$$\boxed{f(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{3} - \cdots\right]}$$ Ans

Parseval's identity.

$$\cancel{\int_0^2 [f(x)]^2\,dx = \frac{2}{2}\sum_{n=1}^{\infty} b_n^2}$$

$$\cancel{\int_0^2 x^2\,dx = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(\ )^{2n}}{n}}$$

(b) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ here $l = 2$

$$a_0 = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^2 = 1$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx = \underbrace{\left.\frac{x\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_0^2}_{0} - \int_0^2 1\cdot\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\,dx$$

$$\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi x}{2}\right)_0^2$$

$$= \frac{4}{n^2\pi^2}\left(\cos n\pi - \cos 0\right)$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{for } n = \text{even} \\ -\frac{8}{n^2\pi^2} & \text{for } n = \text{odd} \end{cases}$$

$$\therefore f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}$$

$$\boxed{f(x) = 1 - \frac{8}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)}$$ Ans

$\Rightarrow$ Using Parseval's identity

$$\int_0^2 \{f(x)\}^2\,dx = \frac{l}{2}\left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2\right]$$

$$\Rightarrow \int_0^2 x^2\,dx = \frac{2}{2}\left[2(1)^2 + \sum_{n=1}^{\infty}\left(-\frac{8}{n^2\pi^2}\right)^2\right]$$

$$\Rightarrow \left(\frac{x^3}{3}\right)_0^2 = 2 + \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \left(\frac{8}{3} - 2\right)\times\frac{\pi^2}{64} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{\pi^2}{96} = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$ Ans