Recurrence Formulae for $P_n(x)$

(I) $nP_n = (2n-1)x P_{n-1} - (n-1)P_{n-2}$

Sol: From generating function of Legendre Polynomial.

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{--- (1)}$$

diff. (1) w.r.t. z

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

or $$(x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

or $$(x-z)(1-2xz+z^2)^{-1/2} = (1-2xz+z^2)\sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

or $$(x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} n z^{n-1} P_n(x) \quad \text{--- (2)}$$

equating the coeff. of z^{n-1} on b.s.

$$xP_{n-1} - P_{n-2} = nP_n - 2x(n-1)P_{n-1} + (n-2)P_{n-2}$$

or $$\boxed{nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}} \quad \text{---- (Ia)}$$

If rst. $n = n+1$ then

$$\boxed{(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}} \quad \text{(Ib)}$$

OR from (2) equate the coff of z^n on b.s.

Note :- $x\sum_{n=0}^{\infty} z^n P_n(x) = 1\cdot\sum_{n=0}^{\infty} n z^{n-1} P_n(x)$ 1st term of eqn (2)

in LHS $n \to n-1$ [because we are equating coeff of z^{n-1} on b.s]

$$xP_{n-1}(x) = nP_n(x)$$

(II) $nP_n = xP_n' - P_{n-1}'$

Sol. From generating function

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{---} \quad (1)$$

diff (1) w.r.t. z

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$\text{or } (x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} n z^{n-1} P_n(x) \quad \text{---} \quad (2)$$

diff (1) w.r.t. x

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2z) = \sum_{n=0}^{\infty} z^n P_n'$$

$$\text{or } z(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n' \quad \text{---} \quad (3)$$

(2)/(3)

$$\frac{x-z}{z} = \frac{\sum_{n=0}^{\infty} n z^{n-1} P_n(x)}{\sum_{n=0}^{\infty} z^n P_n'}$$

$$(x-z)\sum_{n=0}^{\infty} z^n P_n' = z\sum_{n=0}^{\infty} n z^{n-1} P_n(x) \quad \text{---} \quad (4)$$

equating the coeff. of z^n on both sides of (4)

$$xP_n' - P_{n-1}' = nP_n$$

or $\boxed{nP_n = xP_n' - P_{n-1}'}$ (II)

(III) $P'_{n+1} - P'_{n-1} = (2n+1) P_n$

Sol. From relation (Ib) we have

$$(n+1) P_{n+1} = (2n+1) x P_n - n P_{n-1} \quad \text{--- (1)}$$

diff. (1) w.r.t. x

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1) x P_n' - n P'_{n-1}$$

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1) (n P_n + P'_{n-1}) - n P'_{n-1}$$

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1) n P_n + (2n+1) P'_{n-1} - n P'_{n-1}$$

$$(n+1) P'_{n+1} = (2n+1) P_n (1+n) + (2n+1-n) P'_{n-1}$$

[from relation (II) $x P_n' = n P_n + P'_{n-1}$]

collecting the coefficient of P_n & P'_{n-1} we get

$$(n+1) P'_{n+1} = (2n+1)(n+1) P_n + (n+1) P'_{n-1}$$

$$(n+1) P'_{n+1} - (n+1) P'_{n-1} = (n+1)(2n+1) P_n$$

or $$\boxed{P'_{n+1} - P'_{n-1} = (2n+1) P_n} \quad \text{(III)}$$

Put $n = n-1$ then

$$\boxed{P'_n - P'_{n-2} = (2n-1) P_{n-1}} \quad \text{(III b)}$$

(IV) $P'_{n+1} - xP'_n = (n+1) P_n$

From relation (Ib) we have

$$(n+1) P_{n+1} = (2n+1) x P_n - n P_{n-1} \quad \text{------} \quad (1)$$

Diff (1) w.r.to x we get

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1) x P_n' - n P'_{n-1}$$

$$= (2n+1) P_n + (n+1+n) x P_n' - n P'_{n-1}$$

$$= (2n+1) P_n + (n+1) x P_n' + n x P_n' - n P'_{n-1}$$

$$= (2n+1) P_n + (n+1) x P_n' + n (x P_n' - P'_{n-1})$$

$$= (2n+1) P_n + (n+1) x P_n' + n \cdot (n P_n)$$

$$\left[\begin{array}{l} \text{since from} \\ \text{relation (II)} \\ n P_n = x P_n' - P'_{n-1} \end{array} \right]$$

$$= (2n+1) P_n + (n+1) x P_n' + n^2 P_n$$

$$= (2n+1+n^2) P_n + (n+1) x P_n'$$

$$= (n+1)^2 P_n + (n+1) x P_n'$$

$$P'_{n+1} = (n+1) P_n + x P_n'$$

or $P'_{n+1} = x P_n' + (n+1) P_n$

or $\boxed{P'_{n+1} - x P_n' = n+1 P_n}$ — (IV)

If st $n = n-1$ we get

$$\boxed{P'_n - x P'_{n-1} = n P_{n-1}} \quad \text{— (IV b)}$$

Bessel's Differential Equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

$$\text{or } \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$$

this is called Bessel's equation and particular solution of this eqn are called Bessel's function of order n.

Bessel's function of First kind [$J_n(x)$]

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left[1+\frac{(-1)\cdot x^2}{2^2\,1!\,(n+1)}+\frac{(-1)^2\,x^4}{2^4\,2!\,(n+1)(n+2)}+\cdots+\frac{(-1)^r\,x^{2r}}{2^{2r}r!\,(n+1)(n+2)\cdots(n+r)}\right]$$

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\times\left[\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{2^{2r}r!\,(n+1)(n+2)\cdots(n+r)}\right]$$

$$\boxed{J_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}}$$

Bessel's Function of Second kind

$$\boxed{J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}}$$

$$\boxed{J_{-n}(x)=(-1)^n J_n(x)}$$

So General sol. is $\boxed{y=AJ_n(x)+BJ_{-n}(x)}$

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Show that

(i) $J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x$

Sol:- We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{1}{\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2^2 \cdot (n+1)} + \frac{x^4}{2^4 \cdot 2 (n+1)(n+2)} + \cdots \right] \quad \text{------ (1)}$$

Put $n = 1/2$

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2} \Gamma(\frac{1}{2}+1)} \left[1 - \frac{x^2}{2 \cdot 2 (\frac{1}{2}+1)} + \frac{x^4}{2^4 \cdot 2 (\frac{1}{2}+1)(\frac{1}{2}+2)} + \cdots \right]$$

$$= \frac{x^{1/2}}{2^{1/2} \Gamma(\frac{1}{2}+1)} \left[1 - \frac{x^2}{2 \cdot \cancel{2} \cdot \frac{3}{\cancel{2}}} + \frac{x^4}{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2} \cdot \frac{3}{\cancel{2}} \cdot \frac{5}{\cancel{2}}} \right.$$

$$= \frac{x^{1/2}}{2^{1/2} \cdot \frac{1}{2} \Gamma(\frac{1}{2})} \left[1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 5} - \cdots \right]$$

Multiply and divide by x on R.H.S

$$= \frac{x^{1/2}}{2^{-1/2} \sqrt{\pi}} \frac{1}{x} \left[x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$\therefore \boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$$

(ii) $J_{-1/2} = \sqrt{\frac{2}{\pi x}} \cos x$

Sol. We know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

$$J_n(x) = \left[\frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2^2 1!(n+1)} + \frac{x^4}{2^4 2!(n+1)(n+2)} - \cdots\right]\right] \quad \text{------ (1)}$$

put $n = -1/2$ in (1) we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma\frac{1}{2}}\left[1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \cdots\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$\boxed{J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x}$$

(iii) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$

Sol. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ ------ (1)

$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ ------ (2)

squaring & adding of (1) & (2) we have

$$\boxed{[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}}$$

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$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

we know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4\, 2(n+1)(n+2)} \cdots\right] \quad \text{--- (1)}$$

at $n = 3/2$ in (1) we get

$$= \frac{x^{3/2}}{2^{3/2}\Gamma(\frac{3}{2}+1)}\left[1 - \frac{x^2}{2^2(\frac{3}{2}+1)} + \frac{x^4}{2}\right.$$

$$J_{3/2}(x) = \frac{x^{3/2}}{2^{3/2}\Gamma(\frac{3}{2}+1)}\left[1 - \frac{x^2}{2.5} + \frac{x^4}{2.4.5.7} - \frac{x^6}{2.4.6.5.7.9} \cdots\right]$$

Multiply & divide by x^2 on R.H.S

$$= \frac{x^{3/2}}{2^{3/2}\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} \cdot \frac{1}{x^2}\left\{x^2 - \frac{x^4}{2.5} + \frac{x^6}{2.4.5.7} - \frac{x^8}{2.4.6.5.7.9} + \cdots\right\}$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{x^2}{3} - \frac{x^4}{2.3.5} + \frac{x^6}{2.3.4.5.7} - \frac{x^8}{2.3.4.5.6.7.9} \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{2x^2}{3!} - \frac{4x^4}{5!} + \frac{6x^6}{7!} \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{(3-1)}{3!}x^2 - \frac{(5-1)}{5!}x^4 + \frac{(7-1)}{7!}x^6 \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{1}{2!} - \frac{1}{3!}\right)x^2 - \left(\frac{1}{4!} - \frac{1}{5!}\right)x^4 + \left(\frac{1}{6!} - \frac{1}{7!}\right)x^6 \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}\right) + \left(-\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} \cdots\right) + \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) + \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\right]$$

$$\boxed{J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]}$$

$$\frac{x^{3/2}}{2^{3/2}\,\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\,\frac{1}{x^2}$$

$$= \frac{x^{3/2}}{2\, 2^{\frac{1}{2}}\,\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\; x^2}$$

$$= \frac{1}{\frac{1}{2^{\frac{1}{2}}}\sqrt{\pi}\; x^{2-\frac{3}{2}}} = \frac{2^{\frac{1}{2}}}{\sqrt{\pi x}} = \sqrt{\frac{2}{\pi x}}$$

(v) $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[-\sin x - \frac{\cos x}{x}\right]$

Sol: We have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4\,2(n+1)(n+2)} - \cdots\right] \quad \text{--- (1)}$$

multiply (1) of numerator & denominator by $(n+1)$

$$\therefore J_n(x) = \frac{x^n(n+1)}{2^n(n+1)\Gamma(n+1)}\left[1 - \frac{x^2}{2\cdot 2(n+1)} + \frac{x^4}{2\cdot 2^4(n+1)(n+2)} - \cdots\right] \quad \text{--- (2)}$$

or $\Gamma(n+2)$ [since $n\Gamma(n) = \Gamma(n+1)$]

st. in (2) $n = -3/2$

$$J_{-3/2}(x) = \frac{x^{-3/2}\left(-\frac{1}{2}\right)}{2^{-3/2}\,\cancel{\Gamma(n+2)}}\left[1 + \frac{x^2}{2} - \frac{x^4}{2\cdot 4\cdot 1\cdot 1} - \cdots\right]$$

$\sqrt{n+2}$

$$= -\sqrt{\frac{2}{\pi x}}\cdot\frac{1}{x}\left[1 + \frac{x^2}{2} - \frac{x^4}{2\cdot 4} + \cdots\right]$$

$$= -\sqrt{\frac{2}{\pi x}}\cdot\frac{1}{x}\left[1 + x^2\left(\frac{1}{1} - \frac{1}{2}\right) - x^4\left(\frac{1}{6} - \frac{1}{24}\right) + \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[-\frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \frac{1}{x}\left(x^2 - \frac{x^4}{3!} + \cdots\right)\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[-\frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \left(x - \frac{x^3}{3!} + \cdots\right)\right]$$

$$\boxed{J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)}$$

$\sqrt{n+2} = \sqrt{-\frac{3}{2}+2}$

$= \sqrt{\frac{4-3}{2}}$

$= \sqrt{\frac{1}{2}}$

$= \sqrt{\pi}$

[since $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$

& $\Gamma(0) = \infty$]

$\frac{1}{6} - \frac{1}{24} = \frac{4-1}{24}$

$= \frac{3}{24}$

$= \frac{1}{8}$

(2)

$\frac{1}{8} = \frac{1}{6} - \cancel{\frac{1}{8}} = \frac{4-3-1}{24}$

$= \cancel{\frac{4}{24}}$

$= \cancel{\frac{8}{8}}$

$\frac{x^{-3/2}\left(-\frac{1}{2}\right)}{2^{-3/2}\sqrt{-\frac{3}{2}+1}} = -\frac{1}{2}\,\frac{2^{3/2}}{x^{3/2}\sqrt{\pi}}$

$= -\frac{1}{\cancel{2}\,x\,x^{1/2}}\,\frac{2^{3/2}}{\sqrt{\pi}}$

$= -\sqrt{\frac{2}{\pi x}}\,\frac{1}{x}$