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Vector space:

A set V is said to be a vector space over a field F if

1. $(V, +)$ is an abelian group
2. $a \cdot v \in V \ \forall\ a \in F, v \in V$ (Scalar Multiplication)
3. Distributive laws are satisfied
   (i) $a(u+v) = au + av \ \forall\ a \in F, u, v \in V$
   (ii) $(a+b)v = av + bv \ \forall\ a, b \in F, v \in V$.
4. $a(bv) = (ab)v \ \forall\ a, b \in F, v \in V$
5. $1.v = v \ \forall\ v \in V$

$1 \to$ multiplicative identity in F

Subspace:

V is a V.S.

$U \subseteq V$ is called subspace of V over F if U is itself a V.S. over F.

Subspace Test:

$U \subseteq V$ is a subspace of V over F

if (i) $u - u' \in U \quad \forall\, u, u' \in U$

(ii) $au \in U \quad \forall\, a \in F,\ u \in U.$

Linear Combination:

$v_1, v_2, \dots, v_n \in V$

Linear combination of $v_1, v_2, \dots, v_n$ is

$a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ where $a_i \in F,\ 1 \le i \le n.$

Linear Span or Span:

Let $S = \{v_1, v_2, \dots v_n\} \in V$

$$\text{Span } S = \langle v_1, v_2, \dots, v_n \rangle$$

$$= \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n \mid a_i \in F,\ 1 \le i \le n\}.$$

Linearly Dependent and Linearly Independent:

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

trivial solⁿ → $a_1 = a_2 = \cdots = a_n = 0$ → $v_1, v_2, \ldots, v_n$ are L. I.

nontrivial solⁿ → $a_1, a_2, \ldots a_n$ (not all zero) → $v_1, v_2, \ldots, v_n$ are L. D.

Basis:

Let V be a vector space over F. A subset B of V is called a basis for V if

(i) B is linearly independent over F

(ii) B spans V $\quad$ i.e. $\text{Span}(B) = V$

Example:

Show that the set $B = \{(1,0,0), (1,0,1), (1,1,1)\}$ is a basis for R^3 over R.

Solⁿ Let $v_1 = (1,0,0)$, $v_2 = (1,0,1)$, $v_3 = (1,1,1)$

So// let $v_1 = (1,0,0)$, $v_2 = (1,0,1)$, $v_3 = (1,1,1)$.

Now $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

$\Rightarrow a_1(1,0,0) + a_2(1,0,1) + a_3(1,1,1) = 0$

$\Rightarrow (a_1 + a_2 + a_3,\ a_3,\ a_2 + a_3) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} a_1 + a_2 + a_3 &= 0 \\ a_3 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}\right\} \begin{aligned} &\Rightarrow a_1 = 0 \\ &\Rightarrow a_3 = 0 \\ &\Rightarrow a_2 = 0 \end{aligned}$$

$\Rightarrow a_1 = a_2 = a_3 = 0$

$\Rightarrow$ The eqn $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$ has only trivial solution.

$\Rightarrow v_1, v_2, v_3$ are L.I.

$\Rightarrow B$ is linearly independent over F.

Now we need to show that $\mathrm{Span}(B) = R^3$

Let (a,b,c) be any arbitrary element of R^3

let (a, b, c) be any arbitrary element of $\mathbb{R}^3$

and let $a_1 v_1 + a_2 v_2 + a_3 v_3 = (a, b, c)$

$$\Rightarrow a_1(1,0,0) + a_2(1,0,1) + a_3(1,1,1) = (a, b, c)$$

$$\Rightarrow (a_1 + a_2 + a_3,\ a_3,\ a_2 + a_3) = (a, b, c)$$

$$\Rightarrow \left.\begin{aligned} a_1 + a_2 + a_3 &= a \\ a_3 &= b \\ a_2 + a_3 &= c \end{aligned}\right\} \begin{aligned} &\Rightarrow a_1 = a - (c-b) - b \\ &\qquad = a - c \\ &\Rightarrow a_3 = b \\ &\Rightarrow a_2 = c - a_3 = c - b \end{aligned}$$

$\therefore a_1 = a - c,\ a_2 = c - b,\ a_3 = b$

Thus, we can write

$$(a, b, c) = (a-c)v_1 + (c-b)v_2 + b v_3$$

$\Rightarrow \{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$

$\Rightarrow \mathcal{B}$ spans $\mathbb{R}^3$.

$\therefore \mathcal{B}$ is a basis for $\mathbb{R}^3$ over $\mathbb{R}$.