$\therefore$ $\mathcal{B}$ is a basis for $\mathbb{R}^3$ over $\mathbb{R}$.

Remarks:

Empty set is always L.I.

and $\text{Span}(\phi) = \{0\}$

$\Rightarrow$ ϕ is a basis for vector space $\{0\}$.

Dimension:

A vector space that has a basis consisting of n elements is said to have dimension n.

OR

If $\mathcal{B}$ is a basis for vector space V, then $\dim V = |\mathcal{B}| =$ no. of elements in $\mathcal{B}$.

Example:

1. vector space $\{0\}$ have dimension 0.
2. vector space $\mathbb{R}^3$ have dimension 3.

Q. The set $V = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

is a vector space over $\mathbb{R}$. Show that the set $B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for V over $\mathbb{R}$.

Sol.// Let $v_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \{v_1, v_2\}$

<u>B is L.I.</u>

let $a_1 v_1 + a_2 v_2 = 0$

$\Rightarrow a_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 &= 0 \\ a_2 &= 0 \end{aligned}\right\} \Rightarrow a_1 = a_2 = 0.$$

$\Rightarrow$ The eqn $a_1 v_1 + a_2 v_2 = 0$ has only trivial soln.

$\Rightarrow$ $\{v_1, v_2\}$ is L-I.

$\Rightarrow$ B is linearly Independent over R.

<u>B spans V:</u>

Let $\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$ be any arbitrary element of V

and let $a_1 v_1 + a_2 v_2 = \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$

$\Rightarrow a_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_1 & a_1+a_2 \\ a_1+a_2 & a_2 \end{bmatrix} = \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} a_1 &= a \\ a_1 + a_2 &= a+b \\ a_1 + a_2 &= a+b \\ a_2 &= b \end{aligned}\right\} \Rightarrow a_1 = a,\quad a_2 = b$$

$\therefore$ We can write

$$\begin{bmatrix} a & a+b \\ a+b & a \end{bmatrix} = a v_1 + b v_2$$

$\Rightarrow$ $\{v_1, v_2\}$ spans V. $\left(\because \begin{bmatrix} a & a+b \\ a+b & a \end{bmatrix}\right.$ is an arbitrary element of V.

$\Rightarrow$ β spans V

Thus, β is a basis for V over $\mathbb{R}$.

$$\dim V = |\beta| = 2$$