Q1:- The temp. $u(x,t)$ at any point of an infinite bar satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; $-\infty < x < \infty$, $t > 0$ and the initial temp. along the length of the bar is given by

$$u(x,0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Det. the expression for $u(x,t)$.

Sol:- Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ------ (1)

& initial conditions $u(x,0) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ --- (2)

range is $-\infty < x < \infty$, so applying F T to b.s of (1)

$$F\left[\frac{\partial u}{\partial t}\right] = F\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\alpha x}\frac{\partial u}{\partial t}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\alpha x}\frac{\partial^2 u}{\partial x^2}\,dx$$

$\left[\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \right.$

$$\Rightarrow \frac{d}{dt}\bar{u}(\alpha,t) = -\alpha^2 F[u(x,t)] = -\alpha^2 \bar{u}(\alpha,t)$$

$$\Rightarrow \frac{d\bar{u}}{\bar{u}} = -\alpha^2 dt$$

$$\Rightarrow \log \bar{u} = -\alpha^2 t + \log c$$

$$\Rightarrow \log\left(\frac{\bar{u}}{c}\right) = -\alpha^2 t$$

$$\Rightarrow \bar{u} = c e^{-\alpha^2 t}$$

$$\Rightarrow \bar{u}(\alpha,t) = c e^{-\alpha^2 t} \quad \text{--- (3)}$$

c = an arbitrary const.

OR

$$\frac{d}{dt}\bar{u} + \alpha^2 \bar{u} = 0$$

$$(D + \alpha^2)\bar{u} = 0$$

A.E is $m + \alpha^2 = 0$

$$m = -\alpha^2$$

$\therefore$ sol^n is given by

$$\bar{u} = A e^{-\alpha^2 t} \quad \text{--- (3)}$$

put $t=0$ in ③, we obtain

$$\bar{u}(\alpha, 0) = A \text{ or } c \quad \text{as we opt.} \quad \text{------} ④$$

Now taking Eqn. ② $u(x,0) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$

Here we have to find the value of const to 'c'

Now taking the F.T. of eqn ②

$$\Rightarrow \bar{u}(\alpha,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,0)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\alpha x}}{-i\alpha} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\alpha}}{-i\alpha} + \frac{e^{i\alpha}}{i\alpha} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2 \sin\alpha}{\alpha} \quad \text{------} ⑤$$

$|x| < 1$

for $-$ve

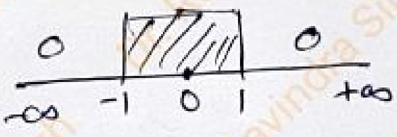
$|x| = -x < 1$

ie $x \geq 1$

for $+$ve

$|x| = x < 1$

$\therefore -1 < x < 1$



from ④ & ⑤ $c = \dfrac{2 \sin\alpha}{\sqrt{2\pi} \cdot \alpha}$

$\therefore$ ③ becomes

$$\bar{u}(\alpha,t) = \frac{2}{\sqrt{2\pi}} \cdot \frac{\sin\alpha}{\alpha}\, e^{-\alpha^2 t}$$

Now taking IFT of above eqn.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\alpha x}\, \bar{u}(\alpha,t)\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sin\alpha}{\alpha}\, e^{-\alpha^2 t}\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin\alpha}{\alpha}\, e^{-\alpha^2 t} \left(\cos\alpha x - i\sin\alpha x \right) d\alpha$$

$$= \frac{2}{\pi} \int_{0}^{\infty} e^{-\alpha^2 t} \left(\frac{\sin\alpha \cos\alpha x}{\alpha} \right) d\alpha - 0 \quad \text{(other is odd)}$$

$$\therefore \boxed{u(x,t) = \frac{2}{\pi} \int_{0}^{\infty} e^{-\alpha^2 t} \left(\frac{\sin\alpha \cos\alpha x}{\alpha} \right) d\alpha}$$

Ans.

10

Q3: Solve $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ for $x \geq 0$, $t \geq 0$ under the given conditions $u = u_0$ at $x = 0$, $t > 0$ with initial conditions $u(x,0) = 0$; $x \geq 0$

Sol^n: Given $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ ---------- (1)

Initial condition $u(x,0) = 0$ ---------- (2)

Boundary condition $u(0,t) = u_0$ ---------- (3)

Now taking FST of eqn (1)

$$\Rightarrow F_s\left[\frac{\partial u}{\partial t}\right] = F_s\left[K\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty K \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx$$

$$\Rightarrow \frac{d \bar{u}_s}{dt} = [\alpha u(x,t)\sqrt{\tfrac{2}{\pi}}_{x=0} - \alpha^2 \bar{u}_s(\alpha, t)] \cdot K$$

$$\frac{d}{dt}\bar{u}_s = \sqrt{\frac{2}{\pi}} K\alpha u_0 - K\alpha^2 \bar{u}_s$$

[as $u(0,t) = u_0$]

$\bar{u}_s$ = FST of u $= F_s[\bar{u}]$

$$\Rightarrow \frac{d}{dt}\bar{u}_s + K\alpha^2 \bar{u}_s = \sqrt{\frac{2}{\pi}} K\alpha u_0 \quad \text{---- (A)}$$

P Q

(A) is linear in $\bar{u}$

$\therefore$ ~~AE is $m + K\alpha^2 = 0$~~

$$IF = e^{K\alpha^2 t}$$

$$\bar{u}_s \times e^{K\alpha^2 t} = \int \sqrt{\frac{2}{\pi}} K\alpha u_0 e^{K\alpha^2 t} dt + C$$

$$\bar{u}_s e^{K\alpha^2 t} = \sqrt{\frac{2}{\pi}} K\alpha u_0 \frac{e^{K\alpha^2 t}}{K\alpha^2} + C$$

$$\bar{u}_s e^{K\alpha^2 t} = \sqrt{\frac{2}{\pi}} \frac{u_0}{\alpha} e^{K\alpha^2 t} + C \quad \text{--- (B)}$$

put $t = 0$

$\left[\frac{dy}{dx} + Py = Q\right]$

$IF = e^{\int P dx}$

Sol^n

$y(IF) = \int Q(I.F)\, dx + C$

$y \times IF = \int Q \times (IF)\, dx + C$

$$\bar{u}_s(\alpha, 0) = \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha} + C$$

as $u(x,0)=0$

so $\bar{u}_s(\alpha,0)=0$

$\bar{u}_s(\alpha,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin\alpha x\,dx$

$\bar{u}_s(\alpha,0) = 0$ at $t=0$

$$\therefore \boxed{C = -\sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha}}$$

eqn (B) gets form

$$\bar{u}_s e^{k\alpha^2 t} = \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha}e^{k\alpha^2 t} - \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha}$$

$$\bar{u}_s = \sqrt{\frac{2}{\pi}}\frac{u_0}{\alpha}\left(1 - e^{-k\alpha^2 t}\right)$$

so Now taking IFST of above eqn we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}_s \sin\alpha x\,d\alpha$$

$$\boxed{u(x,t) = \frac{2}{\pi}u_0\int_0^\infty\left(\frac{1-e^{-k\alpha^2 t}}{\alpha}\right)\sin\alpha x\,d\alpha}$$

Ans

Q5: The temp. 'u' in a semi-infinite rod $0 \le x < \infty$ is determined by by the equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ subject to the conditions

(i) $u = 0$ when $t = 0$, $x \ge 0$ [we can write it as $u(x,0) = 0$]

(ii) $\frac{\partial u}{\partial x} = -\mu$, $\mu \to$ is const. when $x = 0$ & $t > 0$

Determine temp. formula.

Sol^n:- Given $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ —— (1)

initial condition is $u(x,0) = 0$ —— (2)

B.C is $\frac{\partial u}{\partial x} = -\mu$, $t > 0$ & $x = 0$ —— (3)

Taking fourier cosine transform (FCT) of (1) on b-s

$$\Rightarrow F_c\left[\frac{\partial u}{\partial t}\right] = F_c\left[k \frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty k \frac{\partial^2 u}{\partial x^2} \cos \alpha x \, dx$$

$$\Rightarrow \frac{d}{dt} \bar{u}_c(\alpha,t) = k\left(-\sqrt{\frac{2}{\pi}} \frac{\partial u}{\partial x}(x,t)\Big|_{x=0} - \alpha^2 \bar{u}_c(\alpha,t)\right)$$

where $\bar{u}_c(\alpha,t) = F_c[u(x,t)] =$ FCT of $u(x,t)$

$$\Rightarrow \frac{d \bar{u}_c(\alpha,t)}{dt} = \sqrt{\frac{2}{\pi}} k\mu - k\alpha^2 \bar{u}_c(\alpha,t) \qquad \left[\because \frac{\partial u}{\partial x}(0,t) = -\mu\right]$$

$$\Rightarrow \frac{d \bar{u}_c}{dt} + k\alpha^2 \bar{u}_c = \sqrt{\frac{2}{\pi}} k\mu$$

this eqn. is linear in $\bar{u}_c$

$\frac{dy}{dx} + P(x) y = Q$

$P = k\alpha^2$, $Q = \sqrt{\frac{2}{\pi}} k\mu$

$\therefore$ I.F $= e^{\int P dx} = e^{k\alpha^2 t}$

sol^n: $y \times \text{IF} = \int (Q \times \text{IF}) dt + c$

$y e^{k\alpha^2 t} = \int \sqrt{\frac{2}{\pi}} k\mu \cdot e^{k\alpha^2 t} dt + c$

$\therefore$ sol^n is given by

$$\Rightarrow \bar{u}_c e^{k\alpha^2 t} = \int \sqrt{\frac{2}{\pi}} k\mu \cdot e^{k\alpha^2 t} dt + c$$

$$\Rightarrow \bar{u}_c e^{k\alpha^2 t} = \sqrt{\frac{2}{\pi}} k\mu \frac{e^{k\alpha^2 t}}{k\alpha^2} + c$$

$$\bar{u}_c(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2} + c e^{-k\alpha^2 t} \quad \text{---} (4)$$

put $t=0$ in (4)

$$\bar{u}_c(\alpha,0) = \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2} + c \quad \text{---} (5)$$

$$\therefore c = \bar{u}_c(\alpha,0) - \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2} = F_c[u(x,0)] - \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\cos\alpha x\,dx - \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2}$$

$$= -\sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2} \quad \text{as } u(x,0)=0) \text{ by eqn } (2)$$

$\therefore$ eqn (4) becomes

$$\bar{u}_c(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2} - \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2}e^{-k\alpha^2 t}$$

$$= \sqrt{\frac{2}{\pi}}\frac{u}{\alpha^2}\left(1 - e^{-k\alpha^2 t}\right)$$

Now taking IFCT of above eqn. we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}_c(\alpha,t)\cos\alpha x\,d\alpha$$

$$\boxed{u(x,t) = \frac{2u}{\pi}\int_0^\infty \left(\frac{1-e^{-k\alpha^2 t}}{\alpha^2}\right)\cos\alpha x\,d\alpha}$$

It is required solution.

Q7. Solve the eqn. $\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2}$, $x>0$, $t>0$ subject to conditions:

(i) $y = a$ when $x = 0$, $t > 0$

(ii) $y = 0$ when $t = 0$, $x > 0$

Solⁿ: Range of x is $(0 \text{ to } \infty)$

Given $\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2}$ --- (1)

Initial cond $y(x,0) = 0$ --- (2)

B.C. $y(0,t) = a$ --- (3)

Taking FST of (1) on b.s

$\Rightarrow F_s\left[\frac{\partial y}{\partial t}\right] = F_s\left[k\frac{\partial^2 y}{\partial x^2}\right]$

$\Rightarrow \frac{d}{dt}\bar{y}_s(\alpha,t) = \sqrt{\frac{2}{\pi}}k\alpha\, y(0,t) - k\alpha^2\bar{y}_s(\alpha,t)$

where $\bar{y}_s(\alpha,t) = F_s[y(x,t)]$

$\Rightarrow \frac{d}{dt}\bar{y}_s = \sqrt{\frac{2}{\pi}}\alpha k a - k\alpha^2\bar{y}_s$

$\Rightarrow \frac{d}{dt}\bar{y}_s + k\alpha^2\bar{y}_s = \sqrt{\frac{2}{\pi}}k\alpha a$

Eqn is linear in $\bar{y}_s$ like

$\frac{dy}{dx} + Py = Q$

$IF = e^{\int P dx}$

solⁿ, $y \times IF = \int Q \times IF\, dx + C$

Solⁿ: $\bar{y}_s \times e^{k\alpha^2 t} = \int\sqrt{\frac{2}{\pi}}k\alpha a \cdot e^{k\alpha^2 t}dt + C$

$\bar{y}_s e^{k\alpha^2 t} = \sqrt{\frac{2}{\pi}}k\alpha a \cdot \frac{e^{k\alpha^2 t}}{k\alpha^2} + C$

$\bar{y}_s = \sqrt{\frac{2}{\pi}}\frac{a}{\alpha} + Ce^{-k\alpha^2 t}$ --- (4)

put $t = 0$

$\bar{y}(\alpha,0) = \sqrt{\frac{2}{\pi}}\frac{a}{\alpha} + C$ --- (5)

But given $y(x,0)=0$

i.e. $\bar{y}_s(x,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty y(x,0) \sin \alpha x \, dx = 0$ as $y(x,0)=0$

$\therefore$ (5) becomes

$$\boxed{C = -\sqrt{\frac{2}{\pi}} \cdot \frac{a}{\alpha}}$$

putting C in (4) we get

$\therefore$

$$\bar{y}_s(\alpha,t) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{\alpha} - \sqrt{\frac{2}{\pi}} \frac{a}{\alpha} e^{-K\alpha^2 t}$$

Now taking IFST of above eqn. we get

$$\Rightarrow y(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{y}_s(\alpha,t) \sin \alpha x \, d\alpha$$

$$\Rightarrow y(x,t) = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot a \int_0^\infty \left(\frac{1-e^{-K\alpha^2 t}}{\alpha} \right) \sin \alpha x \, d\alpha$$

$$\Rightarrow \boxed{y(x,t) = \frac{2a}{\pi} \int_0^\infty \left(\frac{1-e^{-K\alpha^2 t}}{\alpha} \right) \sin \alpha x \, d\alpha}$$

Ans.