Short Rates:- Short rates are the forward rates spanning a single time period. The short rate at time k is accordingly $r_k = f_{k,k+1}$. i.e. it is the forward rate from k to $k+1$.

$$(1+s_k)^k = (1+r_0)(1+r_1)\cdots(1+r_{k-1}).$$

32
83

In general

$$(1+f_{i,j})^{j-i} = (1+r_i)(1+r_{i+1})\cdots(1+r_{j-1}).$$

Invariance theorem:- Suppose that interest rates evolve according to expectations dynamics. Then (assuming a yearly compounding convention) a sum of money invested in the interest rate market for n years will grow by a factor of $(1+s_n)^n$ independent of the investment and reinvestment strategy (so long as all funds are fully invested).

Ex:- Suppose we have 100, and two interest rates 6.00, 6.45.

→ Option 1:- if invested for two years. $100(1+0.0645)^2$
$= 113.316$

Option 2:- $100(1+0.06)(1+0.069) = 113.314$

Discount factor:-

$$d_{j,k} = \left[\frac{1}{1+f_{j,k}}\right]^{k-j}$$

this factor satisfies

$$d_{i,k} = d_{i,j}\, d_{j,k} \quad \text{for } i<j<k$$

Running Present Value:-

Suppose $(x_0, x_1, x_2, \ldots, x_n)$ is a cash flow stream. We denote the present value of this stream $PV(0)$, meaning the present value at time zero. Now imagine k time periods have passed, then the remainder of cash flow stream is $(x_k, x_{k+1}, \ldots, x_n)$ we can calculate its present value denoted by $PV(k)$

Now,

$$PV(0) = x_0 + d_1 x_1 + d_2 x_2 + \cdots + d_n x_n$$

where d_k's are discount factor at time zero.

$$PV(0) = x_0 + d_1\left[x_1 + (d_2/d_1) x_2 + \cdots + (d_n/d_1) x_n\right]$$

$$= x_0 + d_1 PV(1) \quad (\because d_k/d_1 \text{ are discount factor 1 years from now})$$

In general,

$$PV(k) = x_k + d_{k,k+1} x_{k+1} + d_{k,k+2} x_{k+2} + \cdots + d_{k,n} x_n$$

$$PV(k) = x_k + d_{k,k+1}\left[x_{k+1} + d_{k+1,k+2} x_{k+2} + \cdots + d_{k+1,n} x_n\right]$$

$$PV(k) = x_k + d_{k,k+1}\, PV(k+1).$$

Ex:- (-40, 10, 10, 10, 10, 10, 10) → Cash flow.

(5.0, 5.3, 5.6, 5.8, 6.0, 6.1) → spot rates initial.

5	5.3	5.6	5.8	6.0	6.1
5.6	5.9	6	6.25	6.32	
6.2	6.2	6.467	6.5		
6.2	6.6	6.6			
7	7				
7					

$PV(6) = 10$

$PV(5) = 10 + d_{5,6}(10) = 19.345$

$PV(4) = 10 + d_{4,5}\,PV(5) = 28.080$

$PV(3) = 10 + d_{3,4}(PV(4)) = 36.44$

$PV(2) = 10 + d_{2,3}\,PV(3) = 44.31$

$PV(1) = 10 + d_{1,2}\,PV(2) = 51.963$

$PV(0) = -40 + d_{0,1}\,PV(1) = 9.49$

Floating rate bonds

A floating rate note or bond has a fixed face value and fixed maturity, but its coupon payments are tied to current (short) rate of interest.