Lesson: Boolean Algebra

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BOOLEAN ALGEBRA

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Chapter 2: Boolean Algebra

Boolean Algebra

2.1 Learning outcomes

- Introduction to Boolean Algebra
- To study different Boolean Laws
- Important Boolean Theorems
- Simplification of identities using Boolean Laws and Theorems
- Some interesting facts about Boolean Algebra

2.2 Introduction

Boolean Algebra is just a mathematical expression formed by binary variables and basic logic operations which include OR operation(logical addition), AND operation(logical multiplication, NOT operation(inversion operation). The different variables in normal algebra can take any values from positive to negative, but in the case of Boolean algebra, the variables take only one of the two values 0 or 1. Boolean Laws and theorems help in simplifying various complex logic identities.

2.3 Boolean Laws

Boolean Algebra is a mathematical system which is based on logics. This system has its own fundamental laws.

2.3.1 OR Laws

Law 1 . A+0 = A

If A=0 then 0+0=0
If A=1 then 0+1=1

Law 2. A+1 = A

If A=0 then 0+1=1
If A=1 then 1+1=1

Law 3. A+A=A

If A=0 then 0+0=0
If A=1 then 1+1=1

Law 4. A+A'=1

If A=0, then A'=1 so A+A'=0+1=1
If A=1, then A'=0 so A+A'=1+0=1

For both the cases that is A=0 and A=1, all the four OR laws are satisfied

2.3.2 AND Laws:

Law 1. A.0=A

If A=0 then 0.0=0
If A=1 then 1.0=0

Law 2. A.1=A

If A=0 then 0.1=0
If A=1 then 1.1=1

Law 3. A.A=A

If A=0 then 0.0=0
If A=1 then 1.1=1

Law 4. A.A'=0

If A=0, A'=1 then 0.1=0
If A=1, A'=0 then 1.0=0

We see that for both the cases all the four laws are satisfied

2.3.3 Laws of compliments or Laws not for operation :

These laws just tells about the compliments of the binary numbers.

Law 1. 0'=1

Law 2. 1'=0

Law 3. If A=0, then A'=1

Law 4. If A=1, then A'=0

Law 5. (A')'=A

2.3.4 Commutative Laws :

Law 1. Commutative law for Boolean addition

A+B=B+A

This law states that the order of Boolean addition or in other words the order of OR operation conducted on the variables does not matter.
The following fig 2.3.4 (a) shows the equation:

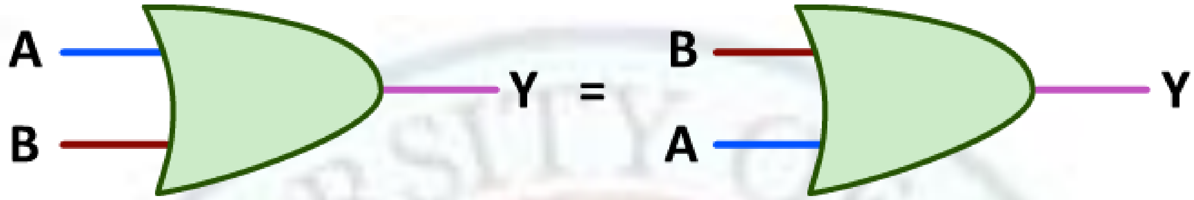


Figure 2.3.4 (a)

Law 2. Commutative law for Boolean multiplication

A.B=B.A

This laws states that the order of Boolean multiplication or the order of AND operation conducted on the variables does not matter.

This is represented by the following fig 2.2.3.4 (b)

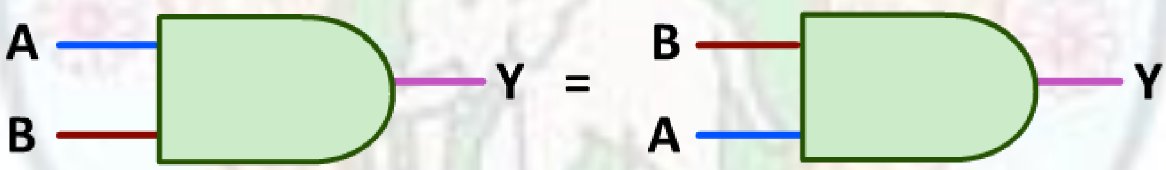


Figure 2.3.4 (b)

2.3.5 Associative Laws:

Law 1. Associative law for Boolean Addition

A+(B+C) = (A+B)+C

This law states that the order of combining variables does not affect the OR operation which is shown in the fig 2.3.5 (a)

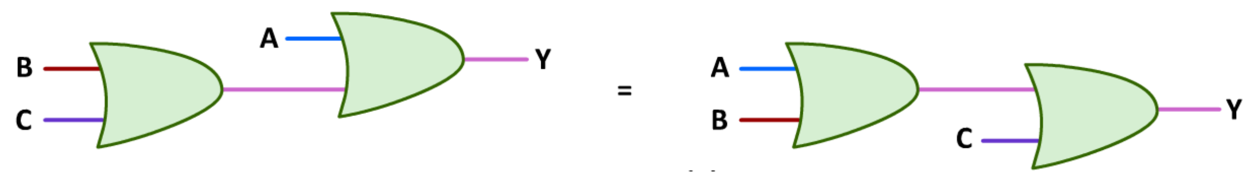


Figure 2.3.5 (a)

Law 2. Associative law for Boolean multiplication

A.(B.C) = (A.B).C

This law states that the order of combining variables does not even affect the AND operation which is shown in the fig 2.3.5 (b)

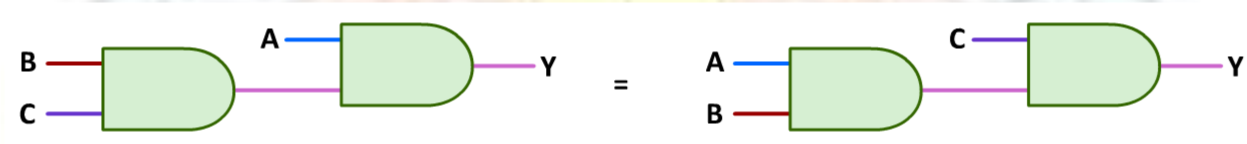


Figure 2.3.5 (b)

2.3.6 Distributive Laws:

Law 1. A.(B+C) = A.B + A.C

This is Boolean multiplication which is distributive over Boolean addition which is shown in the following fig 2.3.6 (a)

Figure 2.3.6 (a)

Law2. A+B.C = (A+B)(A+C)

This is Boolean addition which is distributive over Boolean multiplication

Proof: L.H.S = A+B.C = A.1+B.C (as A.1=A)
= A(1+B)+BC (as 1+B=1)
= A.1+AB+BC (A(B+C)=AB+AC)
= A.(1+C)+AB+BC (1+C=1)
= A.1+AC+AB+BC
= A.A+AC+AB+BC (A.A=A)
= A(A+C)+B(A+C)
= (A+B)(A+C)

2.3.7 Absorptive Laws:

These laws reduce a complicated Boolean expression to a simpler one by absorbing some of the terms into already existing terms.

Law 1. A+AB = A

Proof: L.H.S = A+AB = A.1+AB (A.1=A)
= A(1+B)
= A.1 (1+B=1)
=A=**R.H.S**

Law 2 A+A'B = A+B

Proof : Using the distributive law that A+BC =(A+B)(A+C), we get
L.H.S = A+A'B = (A+A')(A+B)
= 1.(A+B) (A+A'=1)
= A+B = **R.H.S**

LAW 3. A.(A'+B) = AB

Proof: L.H.S = A.(A'+B) = A.A' + A.B (Distributive Law)
= 0+AB (A.A'=0)
=AB = **R.H.S**

2.4 Boolean Theorems:

These are two theorems in Boolean Algebra which play a very important role in simplifying the complicated Boolean expressions.

2.4.1 DeMorgan's Theorem

Two different theorems were proposed by DeMorgan to be used in the simplification. These are:

Theorem 1. (AB)′ = A′ + B′

The first theorem states that the compliment of a product is equal to the sum of the compliments.
This theorem also shows that NAND gate and the bubbled OR gate are equivalent.

Theorem 2. (A+B)′ = A′B′

The second theorem states that the compliment of a sum is equal to the product of the compliments.
This theorem also show that NOR gate and the bubbled AND gate are equivalent.

The compliment of a logical expression can be obtained by the following steps using De Morgan's theorem:

Step1: First of all remove the overall NOT sign

Step 2: Replace all ANDs to ORs and all ORs to ANDs

Step 3: Compliment all the individual variables given in the logical expression.

These two theorems can be proved taking A=0,1 and also B=0,1

When A=0 and B=0, then (AB)′ = (0.0)′ = 0′=1 ; A′=1 and B′=1, then A′+B′=1+1=1:
(A+B)′ = (0+0)′ = 0′ =1; A′B′=1.1=1

When A=0 and B=1, then (AB)′ = (0.1)′ = 0′ =1; A′=1 and B′=0, then A′+B′=1+0=1:
(A+B)′ = (0+1)′=1′=0; A′B′=1.0=0

When A=1 and B=0, then (AB)′=(1.0)′=0′=1; A′=0 and B′=1, then A′+B′=0+1=1:
(A+B)′=(1+0)′=1′=0; A′B′=0.1=0

When A=1 and B=1, then (AB)′=(1.1)′=1′=0; A′=0 and B′=0, then A′+B′=0=0=0:
(A+B)′=(1+1)′=1′=0; A′B′=0.0=0

Thus all the above examples give an authentic proof of the two theorems.

2.4.2 Duality Theorem:

Duality is one of the characteristics of Boolean Algebra. Each expression in Boolean algebra has its dual. The duality theorem is without any proof and states that :
Starting with a Boolean expression, one can get another expression by following the below given steps:

- Change each OR sign to AND sign
- Change each AND sign to OR sign
- Change 0s to 1s and 1s to 0s

For example, consider the relation A+1=1

Its dual relation will be A.0=0 which is obtained by changing OR to AND and by taking compliment of 1 to get 0

Some of the Boolean relations and their duals are given in the following table:

Relation	Dual relation
A.0=0	A+1=1
A.A=A	A+A=A
A.A′=0	A+A′=1
A.1=A	A+0=A
A.(A+B)=A	A+AB=A
A.(A′+B)=AB	A+A′B=A+B

2.6 Simplification of some Boolean Identities or expressions with the help of Boolean Laws and De Morgan Theorem

The above Boolean laws find their application in solving complicated Boolean expressions. We will take up some of the examples.

Example 1. Prove the following Boolean identity, **AB+A′C+BC=AB+A′C**

Proof: AB+A′C+BC = AB+A′C+BC.1
= AB+A′C+BC(A+A′) (A+A′=1)
= AB+A′C+BCA+BCA′
=AB(1+C) + A′C(1+B)
=AB+A′C (1+C=1, 1+B=1)

Example 2. Prove the identity , **(A+B)(A′+C)(B+C)=(A+B)(A′+C)**

Proof: (A+B)(A′+C)(B+C) = (A+B)(A′+C)(B+C+AA′) (AA′=0)
= (A+B)(A′+C)(B+C+A)(B+C+A′)
{Here, we have used the distributive property, A+BC=(A+B)(A+C)}
= (A+B)(A+B+C)(A′+C)(A′+C=B)
= (A+B)(A′+C)
{Here, we have used the absorptive law, A.(A+B)=A

Example 3. Prove the identity, **(A+B)(A+B′)(A′+C)=AC**

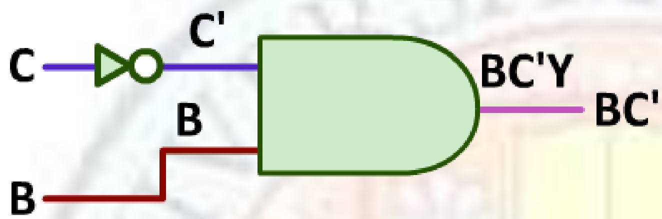
Proof: (A+B)(A+B′)(A′+C) = (AA+AB′+AB+BB′)(A′+C)
= (A+AB+AB′)(A′+C) (BB′=0)
= {A(1+B)+AB′}(A′+C)
= (A+AB′)(A′+C) (1+B′=1)
= A(1+B′)(A′+C)
= A(A′+C)= AA′+AC = AC (AA′=0)

Example 4. Simplify the following identity and give the logic circuit using logic gates:

Y=(ABC′D′)+(A′BC′D′)+(BC′D)

Proof : Y=(A.B.C′.D′)+(A′.B.C′.D′)+(B.C′.D) = BC′D′(A+A′) + (BC′D)
= BC′D′+BC′D
= BC′(D′+D) (D+D′=1)
= BC′

The logic circuit for the above result is:

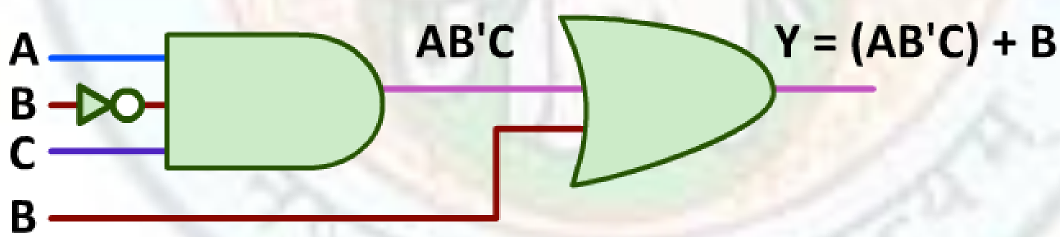


Example 5. Simplify the following identity and give the logic circuit

Y=A′BC+AB′C+ABC+BC′

Proof : Y=(A′BC+AB′C+ABC+BC′) = BC(A+A′) + AB′C + BC′
= BC + AB′C +BC′ (A+A′=1)
= B(C+C′) + AB′C
= B + AB′C

The logic circuit for the above result is:



Example 6. Simplify the following expressions using De Morgan's theorem:

(i) **{A(B+C′)′D}′**
(ii) **{(M+N′)(M′+N)}′**

Proof :

(i) {A(B+C′)′D} = A′+((B+C′)′)′+D′ (as (AB)′=A′+B′)
= A′+B+C′+D′ (as (A′)′=A)

(ii) {(M+N′)(M′+N)}′= (M+N′)′+(M′+N)′
= M′(N′)′+(M′)′N′
= M′N+MN′

Example 7. Simplify the following logic expression

Y=(A′+B)(A+B)

Proof : Y=(A′+B)(A+B) = AA′+A′B+BA+BB
= 0+B(A′+A)+B (AA′=0, BB=B)
= B.1+B (A+A′=1)
= B+B=B

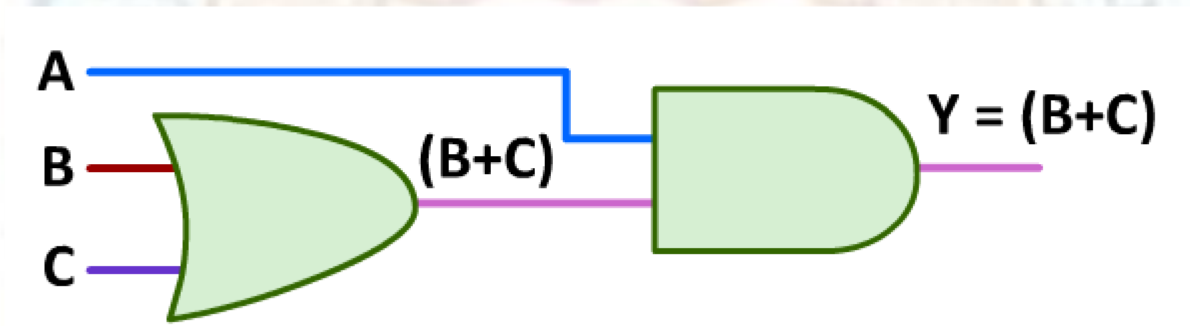
Example 8. Simplify the following expression and draw the logic circuit

Y=ABC+AB′C+ABC′

Proof: Y= ABC+AB′C+ABC′= AC(B+B′)+ABC′
= AC.1+ABC′ (B+B′=1)
=AC+ABC′
=A(C+C′B)
=A(B+C)

We have used the Boolean identity, A+A′B=A+B

The logic circuit for the above result will be



Example 9. Simplify the expression: **Y={(AB′+ABC)′+A(B+AB′)}′**

Proof: Y={(AB′+ABC)′+A(B+AB′)}′=[{A(B′+BC)}′+A(B+A)]′
=[{A(B′+C)}′+AB+A.A]′ (A+A′B=A+B, A′+AB=A′+B)
={(AB′+AC)′+AB+A}′
={(AB′+AC)′+A}′ (A+AB=A)
={(AB′)′.(AC)′+A}′ {(A+B)′=A′.B′}
={(A′+B).(A′+C′)+A}′ {(AB)′=A′+B′}
={A′.A′+A′C′+BA′+BC′+A}′
={A′+A′C′+BA′+BC+A}′
={A′(1+C′)+BA′+BC′+A}′

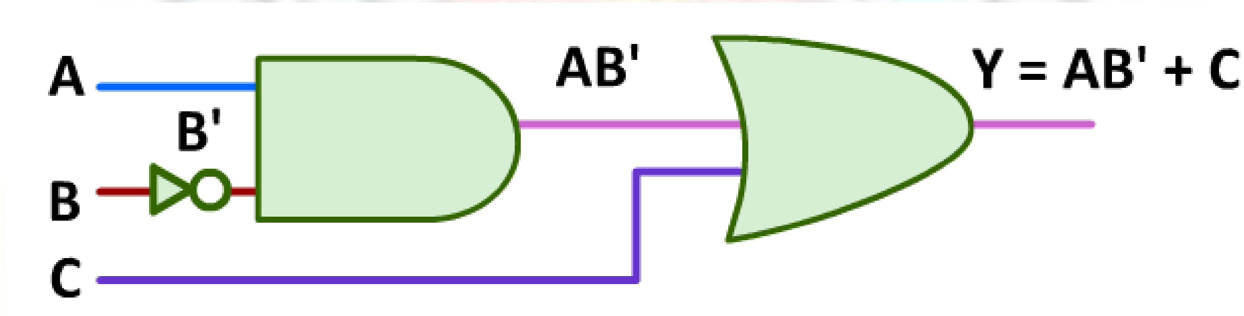
$=\{A'+BA'+BC'+A\}'$
$=\{A'+BC'+A\}' =\{1+BC'\}'$
$=1'=0$

Example 10. Simply the expression and draw its logic circuit

$Y=AB'+(A'+B)C$

Proof: $Y= AB'+(A'+B)C =AB'+\{A'+(B')'\}C$ $\{(B')'=B\}$
$=AB'+(AB')'C$
$=AB'+C$ $(A+A'B=A+B)$

The logic circuit for the above result is



2.6 Summary

- To summarize, we have studied the various Boolean Laws and theorems and their significance in making some very complex identities very simple.
- The Boolean laws include OR laws, AND laws, commutative laws, associative laws, distributive laws etc.
- The major Boolean theorems include DeMorgan's theorem and the Duality theorem.

2.7 Some interesting facts:

- Boolean Algebra is a subarea of algebra in which the values of the variables are the true values denoted by 1 or 0.
- It plays a vital role in the development of digital electronics.
- Boolean Algebra was introduced by an English Mathematician, George Boole in his book , "The Mathematical Analysis of Logic", (1847)
- Boole designed Boolean Algebra for describing and manipulating logical statement and determining if they are true or not.

George Boole

2.8 Exercises:

2.8.1 Fill in the blanks:

1. A + A'B = -----------
2. A(A+B) = ------------
3. A(A'+B) = -----------
4. The dual for the relation A.A=A is --------------
5. The dual for the relation A+A'=1 is ------------

Answers:

1. A+B
2. A
3. AB
4. A+A'=A
5. A.A'=0

2.8.2 Multiple choice questions:

1. The complement of (A+BC+AB) is
   (i) A'(B'+C')
   (ii) A'+B'+C'
   (iii) A'B'C'
   (iv) (A'+B')C'

2. The complement of (A+B)(B+C)(A+C) is
   (i) AB'+BC'+A'C
   (ii) A'B'+B'C'+A'C'
   (iii) AB+BC'+A'C
   (iv) AB+BC+AC'

3. Y=(A+B)(A+C) is equivalent to
   (i) A+BC
   (ii) A'+BC
   (iii) A+B'C

(iv) A+BC’

4. On reducing the following Boolean expression,

   A+A’+B+C

   the result we get is

   (i) 1
   (ii) 0

5. Y=AB+B+A+C is equivalent to
   (i) ABC
   (ii) A’BC
   (iii) A+B+C
   (iv) (A+B)C

Answers:
1. (i)
2. (ii)
3. (i)
4. (i)
5. (iii)

2.8.3 subjective questions :

Q1. State and explain the basic logic operations.

Q2. What are the applications of Boolean Algebra?

Q3. State and explain the two theorems constituting De Morgan’s theorem.

Q4. Discuss the Duality theorem with examples.

Q5. Using Boolean techniques simplify the following expressions and draw the logic circuits for the resultant expressions:

(i) AB+A(B+C)+B(B+C) Ans: B+AC
(ii) AB(C+BD’)+(A’B)’ Ans: (ABC’D)

Q6. If A’B+CD’ = 0, then prove that
AB+C’(A’+D’) = AB+BD+B’D’+A’C’D

Q7. If AB’+A’B=C, then show that
AC’+A’C = B

Q8. Find the compliment of the expression
Y =ABC+ABC’+A’B’C+A’BC

Q9. Indicate whether Y is a 0 or 1 in the expression
Y=AB’C’+AB under the conditions:
(i) A=0, B=1, C=1 Ans: 0
(ii) A=1, B=0, C=1 Ans: 0

Q10. Simplify the following identities :
(i) AB+(AC)′+AB′C (AB+C)
(ii) A′B′C′+A′BC′+AB′C′+ABC′

Q11. Prove the following using De Morgan's theorem:
(i) (A+B)(A′C′+C)(B′+AC)′
(ii) [(AB+C′){(A+B)′+C}]′

Q12. Simplify ABC+AB′C+ABC′ to Y= A(B+C)

Q13. Simplify the expression {(AB′+ABC)′+A(B+AB′)}′

Q14. Prove that A′B′C′+A′B′C+A′BC′+A′BC+AB′C′ = A′+(B+C)′

Q15. Simplify the expression
Y = A′B+ABD+AB′CD′+BC

Answers:

Q5. (i) B+AC

Q9. (i) 0

Q10. (i) 1
(ii)C′

Q11. (i) A′B
(ii)(A′+B′+C′)(A+B+C)

Q13. 0

Q15. A′B+BD+ACD′

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