Present Value and an ideal bank.

An ideal bank can be used to change the pattern of cash flow stream, for example, a 10% bank can change the stream $(1,0,0)$ into the stream $(0,0,1.21)$ by receiving a deposit of 1 now & paying principal and interest of 1.21 in 2 years. In general, if an ideal bank can transform the stream $(x_0, x_1, \ldots, x_n)$ into the stream $(y_0, y_1, \ldots, y_n)$, it can also transform in the reverse direction. Two streams that can be transformed into each other are said to be equivalent streams.

Main theorem on present value :- The cash flow streams $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ are equivalent for a constant ideal bank with interest rate r iff the present values of the two streams, evaluated at bank's interest rate, are equal.

Proof :- Let V_x and V_y be the present values of x and y streams, respectively. Then the x stream is equivalent to the stream $(V_x, 0, \ldots, 0)$ and y stream is equivalent to the stream $(V_y, 0, \ldots, 0)$

Now, it is clear that these two streams are equivalent iff $v_x = v_y$. Hence the original streams are equivalent iff $v_x = v_y$.

Internal rate of return :-

Let $(x_0, x_1, \ldots, x_n)$ be a cash flow stream. Then the internal rate of return of this stream is a number r satisfying the equation

$$0 = x_0 + \frac{x_1}{1+r} + \cdots + \frac{x_n}{(1+r)^n}$$

Equivalently, it is a number r satisfying $1/_{1+r} = c$ where c satisfies the polynomial equation

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots + x_n c^n.$$

Ex:- Consider again the cash flow sequence $(-2, 1, 1, 1)$. The internal rate of return is found by solving

$$0 = -2 + c + c^2 + c^3.$$

then $c = 0.81$, and thus $IRR = 1/c - 1$
$= 0.23.$

Notice that the internal rate of return is defined without reference to a prevailing interest rate. It is defined entirely by the cash flows of the stream. This is the reason why it is called the internal rate of return; it is defined internally without reference to the external financial world. It is the rate that an ideal bank would have to apply to generate the given stream from an initial balance of zero.

<u>Main theorem of IRR</u> :- Suppose the cash flows streams $(x_0, x_1, \ldots, x_n)$ has $x_0 < 0$ and $x_k \geq 0$ for all k; $k = 1, 2, \ldots, n$, with at least one term being strictly positive. Then there is a unique positive root to the equation

$$0 = x_0 + x_1 c + \ldots + x_n c^n$$

furthermore, if $\sum_{k=0}^{n} x_k > 0$ (meaning that the total amount returned exceeds the initial investment), then the corresponding IRR $r = 1/c - 1$ is positive.

<u>Proof</u> :- ~~Note~~ Let $f(c) = x_0 + x_1 c + \ldots + x_n c^n$. Note that $f(0) < 0$. However, as c increases, the value of $f(c)$ increases and since at least one of the cash flow term is strictly positive. Therefore $\lim_{c \to \infty} f(c) = \infty$. Since the function is continuous, it must cross the axis at some value of c. (i.e. $f(c) = 0$).

It cannot cross more than once, because it is strictly increasing. Hence there is a unique real value c_0, which is positive, for which $f(c_0)=0$.

If $\sum_{k=0}^{n} x_k > 0$, which means there is a net positive cash flow, then $f(1) > 0$. $\Rightarrow (f(c) > 0 \ \forall c \geq 1)$ This means that the solution c_0 satisfying $f(c_0)=0$ must be less than 1. Therefore $r_0 = (1/c_0) - 1 > 0$ where r_0 is IRR.

Evaluation Criteria :-

Net Present Value :- Ex:- $(-1, 2)$ cut early
$(-1, 0, 3)$ cut later.

a) $NPV = -1 + \frac{2}{1.1} = 0.82$

b) $NPV = -1 + \frac{3}{(1.1)^2} = 1.48$.

IRR (a) $-1 + 2c \Rightarrow r = 1.0$

(b) $= -1 + 3c^2 \Rightarrow r = \sqrt{3} - 1 \approx 0.7$.

Applications & Extensions.

Net flows :- Net of income minus expenses.

Ex:-

Taxes.

Inflation:- Inflation is characterised by an increase in general prices with time. Inflation can be described quantitatively in terms of an inflation rate f. Prices 1 year from now will on average be equal to today's prices multiplied by (1+f).

· Constant dollars or Real dollars :- These are dollars that continue to have the same purchasing powers as dollars did in the reference year.

Actual or nomial dollars → we use in transactions.

Real interest rate:- It is the rate at which real dollars increase if left in a bank that the nomial rate.

let r is the nomial rate and f is the inflation rate, then

$$1+r_0 = \frac{1+r}{1+f}$$, where r_0 = real interest rate

This equation expresses the fact that money in the bank increases by (1+r), but its purchasing power is deflated by $1/1+f$.

$$r_0 = \frac{r-f}{1+f}$$

Ex:- $f = 4\%$. r (nominal) $= 10\%$.

then $r_0 = 5.77\%$.

Suppose we have a cash flow

(−10,000, 5,000, 5000, 5000, 3000)

~~PV @ 10%~~ [Z10000Z+

Real values

(−10,000, 4807.7, 4622.78, 4444.98, 2564.41)

real PV = |

Ex:- $r = 10\%$ (nominal) $f = 5\%$.

Cash flow = (−100, 220)

$$P.V @ 10\% = -100 + \frac{220}{110} \times 100 = 100$$

Actual flow = (−100, 209.5)

$$P.V = -100 + \frac{209.5}{110} \times 100 = 90.45$$

Exercise question:-

(5) Cash flow (500000, 500000, 5000000, ... $\underbrace{\quad}_{20\ \text{times}}$)

$$PV = 500000 + \frac{500000}{1 + \frac{10}{100}} + \ldots$$

$$= 4,682,460$$

(6). Cash flow 1st Apart

(1000, 1000, 1000, 1000, 1000, 1000)

PV = 1000 + 892.85 + 797.2 + 711.78 + 635.51 + 567.42

= 4604.76

Cash flow 2nd Apm

(1900, 900, 900, 900, 900, 900)

PV = 1900 + 803.57 + 717.47 + 640.6 + 571.96 + 510.68

= 5144.28.

(9) a) Change roof now, then every 20 years.

$$PV_1 = 20{,}000 * \sum_{i=0}^{\infty} \frac{1}{(1.05)^{20i}} = 32{,}097$$

b) Change roof in 5 years, then every 20 years.

$$PV_2 = \frac{PV_1}{(1.05)^2} = 25{,}149.$$

Value of roof = $PV_1 - PV_2$

= 6,948.