Group theory - II Semester - V

Recap:- **Isomorphism :-** An isomorphism ϕ from a group G to a group $\bar{G}$ is one-to-one mapping (or function) from G onto $\bar{G}$ that preserves the group operation, i.e.

$$\phi(ab) = \phi(a)\,\phi(b) \quad \forall\ a, b \in G.$$

If there is an isomorphism from G onto $\bar{G}$, then we say that G is isomorphic to $\bar{G}$.

Example:- (1) Let $G = \mathbb{R}$ (under addition)

$\bar{G} = \mathbb{R}$ (under multiplication)

define $\phi : G \to \bar{G}$ as $\phi(x) = 2^x$.

(2) Any finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

(3) $U(10) \approx \mathbb{Z}_4 \approx U(5)$.

(4) Let $G = SL(2, R) = \{ M \mid \det|M| = 1 \}$

define $\phi_M : G \to G$ as

$$\phi_M(A) = MAM^{-1}$$

then ϕ_M is an isomorphism.

Cayley's theorem :- Every gp is isomorphic to a group of permutations.

Properties of isomorphisms :- $\phi : G \to \bar{G}$ isomorphism

(i) ϕ carries identity of G to the identity of $\bar{G}$.

(ii) For every integer n, and for every $a \in G$.

$$\phi(a^n) = [\phi(a)]^n.$$

(iii) if $ab = ba \iff \phi(a)\phi(b) = \phi(b)\phi(a)$.

(iv) $|a| = |\phi(a)| \quad \forall\, a \in G$.

(v) G is abelian $\iff \bar{G}$ is abelian.

(vi) G is cyclic $\iff \bar{G}$ is cyclic.

(vii) $\phi^{-1} : \bar{G} \to G$ is isomorphism.

(viii) K is subgp of G, then $\phi(K)$ is subgp of $\bar{G}$.

Automorphisms :-

An isomorphism from a group G onto itself is called an automorphism of G.

Set of all automorphisms of G is denoted as $Aut(G)$.

Example :- (i) $\phi : \mathbb{C} \to \mathbb{C}$ under addition.

$$\phi(a+ib) = a - ib$$

& $\phi : \mathbb{C}^* \to \mathbb{C}^*$ under multiplication.

where $\mathbb{C}^*$ is nonzero complex number.

Inner Automorphism Induced by a

Let G be a group, and let $a \in G$. The function ϕ_a defined by $\phi_a(x) = axa^{-1}$ $\forall$ $x \in G$ is called the inner automorphism induced by a.

$Inn(G)$ denotes the set of all inner automorphisms.

Thm:- $Aut(G)$ and $Inn(G)$ are groups.

The set of automorphisms of a group and the set of inner automorphisms of a group are both groups under operation function composition.

Proof:- We will show that $Inn(G)$ is gp.

(i) Let ϕ_a & ϕ_b be inner autom-

then $\phi_a \circ \phi_b = \phi_{ab}$ is also an inner automorphism

(ii) ϕ_e is identity

(iii) $\phi_a \circ (\phi_b \circ \phi_c) = \phi_a \circ (\phi_{bc}) = \phi_{abc}$

$(\phi_a \circ \phi_b) \circ \phi_c = \phi_{ab} \circ \phi_c = \phi_{abc}$.

(iv) for every ϕ_a, $\phi_{a^{-1}}$ is its inverse.

$\therefore$ $Inn(G)$ is a gp.

Example:- (1) $Inn(D_4)$.

ϕ_{R_0}, Let $\phi_{R_{90}}$.

$x \xrightarrow{\phi_{R_{90}}} R_{90}\, x\, R_{90}^{-1}$

$H = -\quad V = |$

$D = \backslash \quad D' = /$

$R_0 \longrightarrow R_{90} R_0 R_{90}^{-1} = R_0$

$R_{90} \longrightarrow = R_{90}$

$R_{180} \longrightarrow R_{180}$

$R_{270} \longrightarrow R_{270}$

$H \longrightarrow V$

$V \longrightarrow H$

$D \longrightarrow D'$

$D' \longrightarrow D$

$R_{90}^{-1} = R_{270}$

$R_{180}^{-1} = R_{180}$

$H^{-1} = H \quad D^{-1} = D$

$V^{-1} = V \quad D'^{-1} = D'$

$\phi_{R_0} = \phi_{R_{180}}$.

$\phi_{R_{90}} = \phi_{270}$

$\phi_H = \phi_V$

$\phi_D = \phi_{D'}$.

$\text{Inn}(D_4) = \{ \phi_{R_0}, \phi_{R_{90}}, \phi_H, \phi_D \}$.

Q.2 $\text{Aut}(Z_{10})$

$Z_{10} = \{0,1,2,3,4,5,6,7,8,9\}$.

$\langle 1 \rangle = \langle 3 \rangle = \langle 7 \rangle = \langle 9 \rangle = Z_{10}$

Let $\alpha \in \text{Aut}(Z_{10}) \Rightarrow \phi : Z_{10} \to Z_{10}$.

if we show $\alpha(1)$ then we can find $\phi(k) = k\phi(1)$.

and since $|\alpha(1)| = 10$

$\Rightarrow \alpha(1) = 1, \quad \alpha(1) = 3, \quad \alpha(1) = 7, \quad \alpha(1) = 9.$

$\quad\ \ \| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \|$

$\quad\ \alpha_1 \qquad\quad\ \alpha_3 \qquad\quad\ \alpha_7 \qquad\quad\ \alpha_9.$

Lt check. α_7.

$\alpha_7(1) = 7$ $\quad \alpha_7(3) = 1$ $\quad \alpha_7(5) = 5$ $\quad \alpha_7(7) = 9$

$\alpha_7(9) = 3$ $\quad \alpha_7(2) = 4$ $\quad \alpha_7(4) = 8$ $\quad \alpha_7(6) = 2$ $\quad \alpha_7(8) = 6$

$$\alpha_7(a+b) = 7(a+b) = 7a + 7b$$
$$= \phi_7(a) + \phi_7(b).$$

$\Rightarrow$ α_7 is a isomorphism.

Illy $\alpha_1, \alpha_3, \alpha_7, \alpha_9 \in Aut(Z_{10})$

$\therefore$ $Aut(Z_{10}) = \{\alpha_1, \alpha_3, \alpha_7, \alpha_9\}$.

$Aut(Z_{10})$	α_1	α_3	α_7	α_9
α_1	α_1	α_3	α_7	α_9
α_3	α_3	α_9	α_1	α_7
α_7	α_7	α_1	α_9	α_3
α_9	α_9	α_7	α_3	α_1

$U(10)$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

Thus! - $\quad Aut(Z_n) \approx U(n)$.