Method of Variation of Parameters:-

Some times, it is very difficult to find particular Integral by using any method to find P.I., for example, It would be difficult to find P.I. of simple equation

$$\frac{d^2y}{dx^2} + y = \tan x$$

We thus seek a method of finding P.I. that applies in all cases in which the complementary function is known. Such a method is the method of Variation of Parameters.

The general second order linear D.Eqn is

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x) \quad \text{--- (1)}$$

Suppose that y_1 and y_2 are two linearly independent solutions of the correspondings homogeneous equations

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad \text{--- (2)}$$

Then the complementary function of equation (1) is

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (2) and c_1 and c_2 are arbitrary constants.

The procedure in the method of variation of parameters is to replace the arbitrary constants c_1 & c_2 in the complementary function by respective functions $v_1(x)$ and $v_2(x)$ which will be determined so that the resulting function, which is defined by

$$v_1(x)y_1(x) + v_2(x)y_2(x), \quad \text{——— (3)}$$

will be particular integral of equation (1).

Now, we assume a solution of the eqn (1) and write

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \quad \text{— (4)}$$

Differentiating (4), we have

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x) + v_1'(x)y_1(x) + v_2'(x)y_2(x)$$

$$\text{— (5)}$$

Now we take

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0 \quad \text{——— (6)}$$

With this condition, eqn (5) reduce to

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) \quad \text{--- (7)}$$

Now differentiating (7), we obtain

$$y_p''(x) = u_1(x)y_1''(x) + u_2(x)y_2''(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x) \quad \text{--- (8)}$$

Since y_p is a solⁿ of (1), thus y_p will satisfy eqn (1), put y_p, y_p' & y_p'' in eqn (1), we get

$$a_0(x)[u_1(x)y_1''(x) + u_2(x)y_2''(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x)]$$
$$+ a_1(x)[u_1(x)y_1'(x) + u_2(x)y_2'(x)]$$
$$+ a_2(x)[u_1(x)y_1(x) + u_2(x)y_2(x)] = F(x)$$

or

$$\cancel{[a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x)]}u_1(x) \rightarrow 0$$
$$+ \cancel{[a_0(x)y_2''(x) + a_1(x)y_2'(x) + a_2(x)y_2'(x)]}u_2 \rightarrow 0$$
$$+ a_0(x)[u_1'(x)y_1'(x) + u_2'(x)y_2'(x)]$$
$$= F(x)$$

we get

$$a_0(x)[u_1'(x)y_1'(x) + u_2'(x)y_2'(x)] = F(x)$$

[Since y_1 & y_2 are L.I. solⁿ of (2),

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or

$$v_1'(x)\, y_1'(x) + v_2'(x)\, y_2'(x) = \frac{f(x)}{a_0(x)} \quad \text{--- (9)}$$

Now solve eqn (8) & (9) or system of eqn's

$$y_1(x)\, v_1'(x) + y_2(x)\, v_2'(x) = 0$$
$$y_1'(x)\, v_1'(x) + y_2'(x)\, v_2'(x) = \frac{f(x)}{a_0(x)}$$

~~we get~~ On solving ~~above~~ above system of eqns, we get

$$v_1'(x) = -\frac{f(x)\, y_2(x)}{a_0(x)\, W[y_1(x), y_2(x)]}$$

&

$$v_2'(x) = \frac{f(x)\, y_1(x)}{a_0(x)\, W[y_1(x), y_2(x)]}$$

where $W[y_1(x), y_2(x)] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

or

$$W[y_1, y_2] = y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)$$

Thus, we obtain the function $v_1(x)$ & $v_2(x)$, as

$$v_1(x) = -\int \frac{f(x)\, y_2(x)}{a_0(x)\, W[y_1(x), y_2(x)]}\, dx$$

$$v_2(x) = \int \frac{f(x)\, y_1(x)}{a_0(x)\, W[y_1(x), y_2(x)]}\, dx \quad \text{--- (10)}$$

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Therefore a P.I. of eqn (1) is given by

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

where u_1 & u_2 are defined by Eqn (10).

Example :- Solve $\dfrac{d^2y}{dx^2} + y = \tan x$ — (1)

The complementary function of (1) is

$$y_c = c_1 \sin x + c_2 \cos x \quad — (2)$$

we assume P.I.

$$y_p(x) = u_1(x) \sin x + u_2(x) \cos x \quad — (3)$$

where, u_1 & u_2 will be determined.

Differentiating eqn (3), we get

$$y_p'(x) = u_1(x) \cos x - u_2(x) \sin x + u_1'(x) \sin x + u_2'(x) \cos x$$

We impose the condition

$$u_1'(x) \sin x + u_2'(x) \cos x = 0 \quad — (4)$$

then, we get

$$y_p'(x) = u_1(x) \cos x - u_2(x) \sin x \quad — (5)$$

Differenting eqn (5), we obtain

(6)

$$y_p''(x) = -u_1(x)\sin x - u_2\cos x + u_1'(x)\cos x - u_2'(x)\sin x \quad \text{—(6)}$$

Substituting value y_p'' & y_p in eq^n (1), we get

$$-\cancel{u_1(x)\sin x} - \cancel{u_2\cos x} + u_1'(x)\cos x - u_2'(x)\sin x$$
$$+ \cancel{u_1(x)\sin x} + \cancel{u_2(x)\cos x} = \tan x$$

or

$$u_1'(x)\cos x - u_2'(x)\sin x = \tan x \quad \text{—(7)}$$

Now, solve eq^n (4) & (7)

$$u_1'(x)\sin x + u_2'(x)\cos x = 0$$
$$u_1'(x)\cos x - u_2'(x)\sin x = \tan x$$

Solving these eq^n, we get

$$u_1'(x) = \sin x$$

& $$u_2'(x) = \cos x - \sec x$$

By integrating these we obtain

$$u_1(x) = -\cos x + C_3$$

& $$u_2(x) = \sin x - \log(\sec x + \tan x) + C_4.$$

then

$$y_p = C_3\sin x + C_4\cos x - \cos x \cdot \log(\sec x + \tan x)$$

[Putting values of u_1 & u_2 in eq^n (3)], Since P.I. is free of constants then assign any particular values A and B to C_3 & C_4.

then

$$P.I. = A\sin x + B\cos x - \cos x \log(\sec x + \tan x)$$

thus general soln of (1)

$$y(x) = y_c + y_p$$

$$y(x) = C_1\sin x + C_2\cos x + A\sin x + B\cos x - \cos x \log(\sec x + \tan x)$$

or

$$\boxed{y(x) = C_1\sin x + C_2\cos x - \cos x \cdot \log(\sec x + \tan x)}$$

& $C_1 = C_1 + A$ & $C_2 = C_2 + B$.

Exercise:- Solve $\frac{d^2y}{dx^2} + y = \sec x$.

Example :- Solve

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2+1)^2 \quad \text{--- (1)}$$

The corresponding homogeneous eq^n is

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \quad \text{--- (2)}$$

The complementary function of (1) or the general sol^n of (2) is

$$y_c(x) = c_1 x + c_2(x^2-1)$$ [we have already solve eq^n (2) in reduction of order section].

To find P.I. of eq^n (1), we assume that

$$y_p = v_1(x)\,x + v_2(x)(x^2-1) \quad \text{--- (3)}$$

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Exercises

Find the general solution of each of the differential equations in Exercises 1–18.

1. $\frac{d^2y}{dx^2} + y = \cot x.$

2. $\frac{d^2y}{dx^2} + y = \tan^2 x.$

3. $\frac{d^2y}{dx^2} + y = \sec x.$

4. $\frac{d^2y}{dx^2} + y = \sec^3 x.$

5. $\frac{d^2y}{dx^2} + 4y = \sec^2 2x.$

6. $\frac{d^2y}{dx^2} + y = \tan x \sec x.$

7. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x} \sec x.$

8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^x \tan 2x.$

9. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \frac{e^{-3x}}{x^3}.$

10. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \ln x \ (x > 0).$

11. $\dfrac{d^2y}{dx^2} + y = \sec x \csc x.$

12. $\dfrac{d^2y}{dx^2} + y = \tan^3 x.$

13. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \dfrac{1}{1 + e^x}.$

14. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \dfrac{1}{1 + e^{2x}}.$

15. $\dfrac{d^2y}{dx^2} + y = \dfrac{1}{1 + \sin x}.$

16. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \sin^{-1} x.$

17. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \dfrac{e^{-x}}{x}.$

18. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x \ln x \quad (x > 0).$

19. Find the general solution of

$$x^2 \frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 10y = 3x^4 + 6x^3,$$

given that $y = x^2$ and $y = x^5$ are linearly independent solutions of the corresponding homogeneous equation.

20. Find the general solution of

$$(x + 1)^2 \frac{d^2y}{dx^2} - 2(x + 1)\frac{dy}{dx} + 2y = 1,$$

given that $y = x + 1$ and $y = (x + 1)^2$ are linearly independent solutions of the corresponding homogeneous equation.

21. Find the general solution of

$$(x^2 + 2x)\frac{d^2y}{dx^2} - 2(x + 1)\frac{dy}{dx} + 2y = (x + 2)^2,$$

given that $y = x + 1$ and $y = x^2$ are linearly independent solutions of the corresponding homogeneous equation.

22. Find the general solution of

$$x^2 \frac{d^2y}{dx^2} - x(x + 2)\frac{dy}{dx} + (x + 2)y = x^3,$$

given that $y = x$ and $y = xe^x$ are linearly independent solutions of the corresponding homogeneous equation.

23. Find the general solution of

$$x(x-2)\frac{d^2y}{dx^2} - (x^2-2)\frac{dy}{dx} + 2(x-1)y = 3x^2(x-2)^2e^x,$$

given that $y = e^x$ and $y = x^2$ are linearly independent solutions of the corresponding homogeneous equation.

24. Find the general solution of

$$(2x+1)(x+1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = (2x+1)^2,$$

given that $y = x$ and $y = (x+1)^{-1}$ are linearly independent solutions of the corresponding homogeneous equation.

25. Find the general solution of

$$(\sin^2 x)\frac{d^2y}{dx^2} - 2\sin x\cos x\frac{dy}{dx} + (\cos^2 x + 1)y = \sin^3 x,$$

given that $y = \sin x$ and $y = x \sin x$ are linearly independent solutions of the corresponding homogeneous equation.

26. Find the general solution by two methods:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = x^2e^x.$$

Solve

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^x$$