Integrating factor :-

If the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{--- (1)}$$

is not exact in a domain D but the differential equation

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0 \quad \text{--- (2)}$$

is exact in D, then $\mu(x,y)$ is called an integrating factor of the differential equation (1).

Example:- $(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$ --- (1)

$M(x,y) = 3y + 4xy^2$ & $N = 2x + 3x^2y$

$\frac{\partial M}{\partial y} = 3 + 8x$, $\frac{\partial N}{\partial x} = 2 + 6xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ equation (1) is not exact.

Now take $\mu(x,y) = x^2y$ and multiply equation (1) by x^2y, then

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0 \quad \text{--- (2)}$$

$M'(x,y) = \mu(x,y) \times M(x,y)$ & $N'(x,y) = \mu(x,y) \times N(x,y)$

$= (3x^2y^2 + 4x^3y^3)$, $= 2x^3y + 3x^4y^2$

$\frac{\partial M'}{\partial y} = 6x^2y + 12x^3y^2$ & $\frac{\partial N'}{\partial x} = 6x^2y + 12x^3y^2$

& $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ $\Rightarrow$ equation (2) is exact.

Hence $\mu(x,y) = x^2y$ is an integrating factor of (1).

Rules to find Integrating factors (I.F.):→

Rule I:- If the equation $M\,dx + N\,dy = 0$ — (1) is homogeneous and $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an I.F. of (1).

Rule II:- If the equation $M\,dx + N\,dy = 0$ can be written as $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx-Ny}$ is an I.F. of (1).

Example:- $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ — (1)

$M = (x^2y - 2xy^2)$, $N = -(x^3 - 3x^2y)$

$\frac{\partial M}{\partial y} = x^2 - 4xy$, $\frac{\partial N}{\partial x} = -3x^2 + 6xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ⇒ eqn (1) is not exact.

$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2$

$Mx + Ny = x^2y^2 \neq 0$

and eqn (1) is ~~is~~ Homogeneous, then I.F of equation (1) is

$$\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

Multiply eqn (1) by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{— (2)}$$

Now $M = \frac{1}{y} - \frac{2}{x}$, $N = -\frac{x}{y^2} + \frac{3}{y}$

$\frac{\partial M}{\partial y} = -\frac{1}{y^2}$, $\frac{\partial N}{\partial x} = -\frac{1}{y^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ eqn (2) is exact.

Soln of equation (2)

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y}\, dy = \text{Constant}$$

[Treat y as Constant]

$$\boxed{\frac{x}{y} - 2\log x + 3\log y = \log C} \quad (\text{Constant})$$

$$\frac{x}{y} + \log \frac{y^3}{x^2} = \log C$$

(you can leave here also)

$$\log C - \log \frac{y^3}{x^2} = \frac{x}{y}$$

$$\log\left(C\,\frac{x^2}{y^3}\right) = \frac{x}{y}$$

$$\boxed{C x^2 = y^3 e^{x/y}}$$

Exercise. Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$

[Hint: Try to use Rule II).

Rule III: If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)dx}$ is an I.f. of $M\,dx + N\,dy = 0$.

Rule IV: If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ (function of y alone), then $e^{\int g(y)dy}$ is an I.f. of $M\,dx + N\,dy = 0$.

Example $(x^2+y^2+2x)\,dx + 2y\,dy = 0$ —— (1)

$M = x^2+y^2+2x$, $N = 2y$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ eqn (1) is not exact.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y}(2y-0) = 1 = f(x)$ (say)

then I.f. of (1) $= e^{\int 1\,dx} = e^x$.

Multiply eqn (1) by e^x, we have

$(x^2+y^2+2x)e^x\,dx + 2ye^x\,dy = 0$ —— (2)

$M_1 = (x^2+y^2+2x)e^x$, $N_1 = 2ye^x$

$\frac{\partial M_1}{\partial y} = 2ye^x$, $\frac{\partial N_1}{\partial x} = 2ye^x$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\Rightarrow$ eqn (2) is exact.

Soln of (1)

$$\int (x^2+y^2+2x)\, e^x\, dx + \cancel{\int 2y e^x dy} = C$$

[Treat y as constant] [Since $N_1(x,y) = 2ye^x$]]

$$(x^2+y^2+2x)\, e^x - \int (2x+2)\, e^x\, dx = C$$

$$(x^2+y^2+2x)\, e^x - (2x+2)\, e^x + \int 2 e^x\, dx = C$$

$$(x^2+y^2+2x)\, e^x - (2x+2)\, e^x + 2e^x = C$$

$$(x^2+y^2+2x-2x-2+2)\, e^x = C$$

$$\boxed{(x^2+y^2)\, e^x = C}$$

Exercise:- (1) $(x^2+y^2)\,dx - 2xy\,dy = 0$ [Hint: Use Rule III]

(2) $(3x^2y^4+2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

[Hint: Use Rule IV]

(3) $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

[Hint: Use Rule IV]

Rule V: If the given D.E. is $Mdx + Ndy = 0$ —(1) is of the form

$$x^{\alpha}y^{\beta}(my\,dx + nx\,dy) = 0,$$ then

$x^{km-1-\alpha}\,y^{kn-1-\beta}$ is I.F. of (1). when m & n are integers and k has any value.

Remark: When an equation can put in the form

$$x^{\alpha}y^{\beta}(my\,dx + nx\,dy) + x^{\alpha_1}y^{\beta_1}(m_1y\,dx + n_1x\,dy) = 0$$

an integrating factor can easily obtained. A factor that will make $x^{\alpha}y^{\beta}(my\,dx + nx\,dy)$ an exact differential is $x^{km-1-\alpha}\,y^{kn-1-\beta}$, when k has any value; and a factor that will make $x^{\alpha_1}y^{\beta_1}(m_1y\,dx + n_1x\,dy)$ an exact differential is $x^{k_1m_1-1-\alpha_1}\,y^{k_1n_1-1-\beta_1}$, k_1 has any value.

these two factors are identical if

$$km - 1 - \alpha = k_1m_1 - 1 - \alpha_1$$

&

$$kn - 1 - \beta = k_1n_1 - 1 - \beta_1.$$

Values of k & k_1 can be found to solve above two linear equations.

Example: Solve $(y^3-2yx^2)dx+(2xy^2-x^3)dy=0$ —(1)

[not exact]

Rewrite

$$y^2(y\,dx+2x\,dy)-x^2(2y\,dx+x\,dy)=0 \quad \text{—(1)}$$

for first term $\alpha=0, \beta=2, m=1, n=2$ and hence

$x^{k-1}y^{2k-1-2}$ is integrating factor.

and for second term $\alpha_1=2, \beta_1=0, m_1=2, n_1=1$ and hence

$x^{2k_1-1-2}y^{k_1-1}$ is integrating factor.

These two integrating factors are same if

$$k-1=2k_1-1-2 \quad \& \quad 2k-1-2=k_1-1$$

On solving these linear equations for k and k_1,

we get $k=2$, & $k_1=2$.

therefore xy is common I.F for both term.

Multiply eqn (1) by xy, we get

$$(xy^4-2x^3y^2)dx+(2x^2y^3-x^4y)dy=0 \quad \text{—(3)}$$

$M=xy^4-2x^3y^2$, $N=2x^2y^3-x^4y$

$\frac{\partial M}{\partial y}=4xy^3-4x^3y$, $\frac{\partial N}{\partial x}=4xy^3-4x^3y$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ eqn (3) is exact.

Soln of (1)

$$\int(xy^4-2x^3y^2)\,dx+\int 0\,dy=c$$

[Treat y as constant]

$$\frac{x^2y^4}{2}-\frac{x^4y^2}{2}=c \Rightarrow \boxed{x^2y^2(y^2-x^2)=c'}$$

Exercise: (1) $(2x^2y-3y^4)\,dx+(3x^3+2xy^3)\,dy=0$

(2) $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0$ (use Rule V)

(3) $(5xy+4y^2+1)\,dx+(x^2+2xy)\,dy=0$

(4) $(2x+\tan y)\,dx+(x-x^2\tan y)\,dy=0$

(5) $[y^2(x+1)+y]\,dx+(2xy+1)\,dy=0$

(6) $(2xy^2+y)\,dx+(2y^3-x)\,dy=0$

(7) $(4xy^2+6y)\,dx+(5x^2y+8x)\,dy=0$

(8) $(8x^2y^3-2y^4)\,dx+(5x^3y^2-8xy^3)\,dy=0$