Classification of first-order PDE :-

The most general, first-order PDE in two independent variables x and y is

$$F(x, y, u, u_x, u_y) = 0 \quad (x, y) \in D \subset R^2 \quad \text{--- (1)}$$

where F is a given function of its arguments, and $u = u(x, y)$ is an unknown function of the independent variables x and y which lie in some given domain D in R^2,
and standard notation $p = u_x$ & $q = u_y$
so that eqn (1) take the form

$$F(x, y, u, p, q) = 0 \quad \text{--- (2)}$$

~~eqn (1) or (2) is called~~

1. Quasi-linear partial differential equation :-

A PdE of first order is called quasi-linear pde if it is linear in first order partial derivatives of unknown function $u(x, y)$. So, the most general quasi-linear pde is

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u)$$

Where a, b, & c are functions of x, y & u, Examples:- (1) $x(x^2+4)u_x - y(x^2+4)u_y = (x^2-y^2)u$

(2) $(y^2-u^2)u_x - xy\,u_y = xu.$

2. Semilinear PDE :-

A first order PDE is called semilinear if its coefficients (a & b) independent of u. Hence semilinear equation can be expressed in the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

Examples:- (1) $xu_x + yu_y = u^2 + x^2$

(2) $(x+1)^2u_x - (y-1)^2u_y = (x+y)u^2$

3. Linear PDE :-

A first order PDE is called linear if it is linear in each of the variables u, u_x & u_y. and the coefficients of these variables are functions only of the independent variables x and y. The most general form of first order linear PDE is

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)$$

Examples: ① $xu_x + yu_y - nu = 0$

② $(y-z)u_x + (z-x)u_y + (x-y)u_z = 0$

Non-linear PDE :-

An equation with is not linear is called non-linear PDE.

Example:- ① $u_x^2 + u_y^2 = 1$

② $u_x \cdot u_y = u$