Article 46 Cauchy - Goursat theorem

Let C denote simple closed contour $z = z(t)$ where $(a \le t \le b)$ in positive sense (counter clockwise) and let f be analytic at each point interior to and on C.

we know, that

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt. \quad -(1)$$

where $f(z) = u(x,t) + iv(x,y)$

$z(t) = x(t) + iy(t)$

then equation (1), can write as

$$\int_C f(z)\,dz = \int_a^b [u(x,y) + iv(x,y)]\,(x'(t) + iy'(t))\,dt$$

$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt$$

Now replacing $f(z)$ & dz on left with $u+iv$ and $dx+idy$ respectively, we get

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C udx - vdy + i\int_C vdx + udy \quad -(2)$$

Equation (2) is also valid when C is any contour, not necessarily a simple closed contour, when $f(z(t))$ is only piecewise continuous on it.

Using Green's theorem

Suppose two real valued function $P(x,y)$ & $Q(x,y)$ together with first order partial derivatives are cts throughout closed region R consisting of all points interior to and on simple closed contour C

then $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy$]

$\therefore$ Using Green's theorem, we have

Now f is cts on R, since it is analytic.

Hence the function u & v are also cts in R.

If f' is cts in R, first order partial derivatives of u & v are also cts.

By applying Green's theorem.

eqn (2), can be write as

$$\int_C f(z)\,dz = \iint_R (-V_x - U_y)\,dx\,dy + i\iint_R (U_x - V_y)\,dx\,dy \quad — (3)$$

But C.R equation is also satisfy $U_x = V_y$

$U_y = -V_x$

$\therefore$ eqn (3), becomes

$$\int_C f(z)\,dz = 0$$

Therefore, when f is analytic at all points interior to and on simple closed contour C and f' is cts

we get $\int_C f(z)\,dz = 0$

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Example) If C is any simple closed contour, in either direction then show $\int_C \exp(z^3)\,dz = 0$

(Solution) Using above result,
Since $f(z) = \exp(z^3)$ is composite function
$\therefore$ it is analytic everywhere
and $f'(z) = 3z^2(\exp(z^3))$ is cts everywhere
$\therefore$ we get $\int_C f(z)\,dz = 0$

Statement of Cauchy - Goursat theorem

Theorem : If function f is analytic at all points interior to and on simple closed contour C, then [proof is Not in syllabus]

$$\int_C f(z)\,dz = 0$$

Remark. [In Cauchy - Goursat theorem, we don't need to assume f' is also cts only need that f' is exists]

(Exercise)
(Q-1) Apply Cauchy- Goursat theorem to show that $\int_C f(z)\,dz = 0$ where Contour C is unit circle $|z| = 1$, in either direction

(a) $f(z) = \dfrac{z^3}{z-3}$

(b) $f(z) = ze^{-z}$

(c) $f(z) = \dfrac{1}{z^2+2z+2}$

(d) $f(z) = \tan z$

(Solution) (a) $f(z) = \dfrac{z^3}{z-3}$

we have $f(z)$ is analytic except at $z-3=0$
$\Rightarrow z=3$

Here C is unit circle $|z|=1$

Now $z=3$

we have $|z| = |3| > 1$

$z=3$ lies outside

$\therefore$ By Cauchy Goursat theorem.

$$\int_C f(z)\,dz = \int_C \frac{z^3}{z-3} = 0$$

In Cauchy-Goursat theorem function is to analytic inside & on C

[Here $f(z) = \dfrac{z^3}{z-3}$ is analytic in & on $|z|=1$

because $z=3$ lies outside $|z|=1$]

(b) $f(z) = ze^{-z} = \dfrac{z}{e^z}$

$f(z)$ is analytic everywhere

Since $\dfrac{z}{e^z}$ is analytic everywhere as $e^z \neq 0$ $\forall z \in \mathbb{C}$

$\exp(z)$ is analytic & z is also analytic

$\therefore f(z)$ is analytic on $|z|=1$

$\therefore$ By Cauchy Goursat theorem,

$$\int_C f(z)\,dz = 0$$

(c) & (d) (do as exercise)

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(5) Couchy Integral formula

Theorem: let f be analytic everywhere inside & on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C then (1)

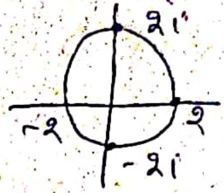
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0} \quad -(1)$$

This called (formula (1)) Cauchy Integral formula.

(1) can be rewritten as

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i\, f(z_0) \quad -(2)$$

(Example) let C be positively oriented circle $|z| = 2$.



find $\displaystyle\int_C \frac{z\,dz}{(9-z^2)(z+i)}$

$z^2 = 9$ $\quad |z| = 2$

$z = \pm 3$ $\quad |3| = 2$

$z = (-i)$ outside

$|-i| = 1 < 2$

(Sol) Since $f(z) = \dfrac{z}{9-z^2}$ is analytic within and on C

(Since $|3| > 2$ outside so, no problem, we check inside & on C)

Since the point $z_0 = -i$ is interior to C

$$\therefore \int_C \frac{z\,dz}{(9-z^2)(z+i)} = \int_C \frac{z/(9-z^2)}{z-(-i)}\,dz$$

As formula $\displaystyle\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i\, f(z_0)$

we get $\displaystyle\int_C \frac{z/(9-z^2)}{z-(-i)}\,dz = 2\pi i\, f(z_0)$

$$= 2\pi i\, f(-i)$$

$$= 2\pi i\, \frac{(-i)}{9-(-i)^2}$$

$$= 2\pi i\, \frac{(-i)}{9+1} = \frac{\pi}{5}$$

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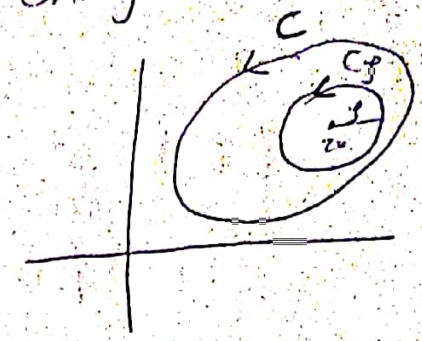
Now proof of Cauchy Integral formula

let C_ρ denote a positively oriented circle $|z-z_0| = \rho$ where ρ is small enough that C_ρ is interior to C.



Since the quotient $\frac{f(z)}{(z-z_0)}$ is analytic between and on the contour C_ρ and C.

(given $f(z)$ be analytic everywhere inside & on simple closed contour C.

($\frac{f(z)}{(z-z_0)}$ is analytic except at $z = z_0$ on C_ρ

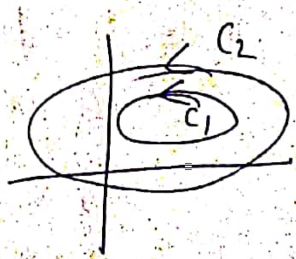
$|z-z_0| = \rho > 0$

$\Rightarrow |z_0 - z_0| = |0| < \rho$)

(Principle of deformation of paths).

let C_1 & C_2 denote positively oriented simple closed contours, where C_1 is interior of C_2 if a function f is analytic in closed region consisting of those contours & all points between them

then $\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz$



$\therefore \int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz$

$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz - f(z_0) \int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z-z_0}\,dz$

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Now, $\displaystyle\int_{C_\rho} \frac{dz}{z-z_0}$

As $C_\rho : z = z_0 + \rho e^{i\theta}, \ 0 \le \theta \le 2\pi$ $\qquad |z-z_0| = \rho$

$z = z_0 + \rho e^{i\theta}$

$dz = \rho e^{i\theta} i \, d\theta$

$$\Rightarrow \int_{C_\rho} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{\rho e^{i\theta} i \, d\theta}{\rho e^{i\theta}} = \int_0^{2\pi} i \, d\theta = 2\pi i \quad \text{(A)}$$

∴ we have

$$\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\, dz \quad \text{—(3)}$$

As f is analytic $\Rightarrow$ f is cts at z_0

∴ for $\epsilon > 0$, $\exists \ \delta > 0$ s.t. (choose)

$|f(z) - f(z_0)| < \frac{\epsilon}{\rho}$ whenever $|z-z_0| < \delta$

Now let radius ρ of circle C_ρ be smaller than the number δ

i.e. $|z-z_0| = \rho < \delta$

$$\therefore \left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| \le \int_{C_\rho} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| dz$$

$$< \frac{\epsilon}{\rho} \int_{C_\rho} \frac{dz}{|z-z_0|}$$ (A) above as $|z-z_0|$

$$= \frac{\epsilon}{\rho}\, 2\pi \rho$$ $\qquad z - z_0 = \rho e^{i\theta}$, $|z-z_0| = \rho$

$$= 2\pi\epsilon$$

∴ from eqn (3), we get

$$\left| \int_C \frac{f(z)}{z-z_0}\, dz - 2\pi i f(z_0) \right| < 2\pi\epsilon \quad \forall \epsilon > 0$$

(choose small as small)

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$$\left| \int_C \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| < 2\pi \epsilon \quad \forall \epsilon > 0$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

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