Charpit's Method :-

Working rule, while using Charpit's Method :-

Step (I) Write the given eqn in the form
$$f(x, y, z, p, q) = 0$$

(II) Write Charpit's equation or Charpit's Auxiliary eqn.

$$\boxed{\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q} = \frac{dp}{-(f_x + p f_z)} = \frac{dq}{-(f_y + q f_z)}}$$

(III) Select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least p & q.

(IV) The simplest relation of step III is solved along with the given eqn to determine p & q. Put these values of p & q in $dz = p\,dx + q\,dy$, which on integration gives the complete integral of the given equation.

Example :- find a complete integral of the equation

$$p^2 x + q^2 y = z \quad \text{—— (I)}$$

Let-

$$f \equiv p^2 x + q^2 y - z = 0 \quad \text{—— (2)}$$

The Charpit's Auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{(p f_p + q f_q)} = \frac{dp}{-(f_x + p f_z)} = \frac{dq}{-(f_y + q f_z)}$$

or

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2 x + q^2 y)} = \frac{dp}{-(p^2 - p)} = \frac{dq}{-(q^2 - q)}$$

or

$$\boxed{\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2 x + q^2 y)} = \frac{dp}{p - p^2} = \frac{dq}{q - q^2}} \quad \text{—— (3)}$$

Taking last two fractions of above eqn, we get

$$\frac{dp}{p - p^2} = \frac{dq}{q - q^2} \quad \text{or} \quad \frac{dp}{p^2 - p} = \frac{dq}{q^2 - q}$$

or

$$\boxed{\frac{dp}{p(p-1)} = \frac{dq}{q(q-1)}}$$

(3)

From function of eqn (3)

$$\frac{p^2 dx + 2px\,dp}{\cancel{2p^3x} + 2p^2x - \cancel{2p^3x}} = \frac{q^2 dy + 2qy\,dq}{\cancel{2q^3y} + 2q^2y - \cancel{2q^3y}}$$

or

$$\frac{p^2 dx + 2px\,dp}{2p^2x} = \frac{q^2 dy + 2qy\,dq}{2q^2y}$$

$$\frac{1}{p^2x}\,d(p^2x) = \frac{1}{q^2y}\,d(q^2y)$$

on Integrating

$$\log(p^2x) = \log(q^2y) + \log a$$

or $\boxed{p^2x = aq^2y}$ —— (4)

where 'a' is a constant.

Now solving eqn (1) & (4), [Putting $p^2x = aq^2y$ in eqn (1)]

we get

$$aq^2y + q^2y = z$$

or $$q^2 = \frac{z}{(1+a)y}$$

or $$q = \left[\frac{z}{(1+a)y}\right]^{1/2}$$

and similarly

$$p = \left[\frac{az}{(1+a)x}\right]^{1/2}.$$

and we know

$$dz = p\,dx + q\,dy$$

$$dz = \left[\frac{az}{(1+a)x}\right]^{1/2} dx + \left[\frac{z}{(1+a)y}\right]^{1/2} dy$$

or

$$\left(\frac{1+a}{z}\right)^{1/2} dz = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{1}{y}\right)^{1/2} dy$$

on Integrating

$$2\,(1+a)^{1/2} \cdot z^{1/2} = 2a^{1/2}\,x^{1/2} + 2\,y^{1/2} + 2b.$$

or

$$\boxed{[(1+a)z]^{1/2} = (ax)^{1/2} + y^{1/2} + b}$$

is Complete integral of (1)

when b is a Constant.

Exercises:- Find the complete integrals of the following eq^{ns} :-

(1) $(p^2+q^2)y = qz$

(2) $p = (z+qy)^2$

(3) $z^2 = pqxy$

(4) $xp+3yq = 2(z-x^2q^2)$

(5) $px^5-4q^3x^2+6x^2z-2 = 0$

(6) $2(y+zq) = q(xp+yq)$

(7) $2(z+xp+yq) = yp^2$