Discrete Mathematics

Defn 1 A relation R on a set A is called Reflexive if $aRa \ \forall a \in A$
i.e. $(a,a) \in R \ \forall a \in A$

Defn 2 A relation R on a set A is called antisymmetric if whenever aRb & bRa then $a = b$.

Defn 3 A relation R on a set A is called transitive if whenever aRb & bRc then aRc.

Defn 4 Partial Order Relation.
A relation R on a set A is called a Partial Order relation if R is
1) Reflexive
2) Antisymmetric
3) Transitive.

A partial order relation is usually denoted by '$\leq$'.

ex 1 $\mathbb{N}$ = natural nos.
$\leq$ is the usual order on $\mathbb{N}$.
'$\leq$' is a partial order relation on $\mathbb{N}$.

Defn If $\leq$ is a partial order relation on a set P, then

$(P, \leq)$ is called a Partially Ordered Set or a POSET.

ex 2 $\mathbb{N}$
Let $a \leq b$ iff a divides b $(a|b)$
Then $(\mathbb{N}, \leq)$ is a Poset

Ex 3 P = set of all factors of 54
Let $a \leq b$ in P iff $a|b$
Then $(P, \leq)$ is a Poset.

ex 4 Let $A = \{1, 2, 3\}$
Let $\mathcal{P}(A)$ = power set of A.
Define $X \leq Y$ in $\mathcal{P}(A)$ iff $X \subseteq Y$.

Then $(\mathcal{P}(A), \subseteq)$ is a Poset.

Defn A poset $(P, \leq)$ is called a chain (or a linearly ordered set or a Totally ordered set or a TOSET) if $\forall a, b \in P$, Either $a \leq b$ or $b \leq a$.

ex $(\mathbb{N}, \leq)$ is a chain.

Defn 2 elements a & $b \in P$ (a Poset) are s.t. b Comparable if Either $a \leq b$ or $b \leq a$.

Remark

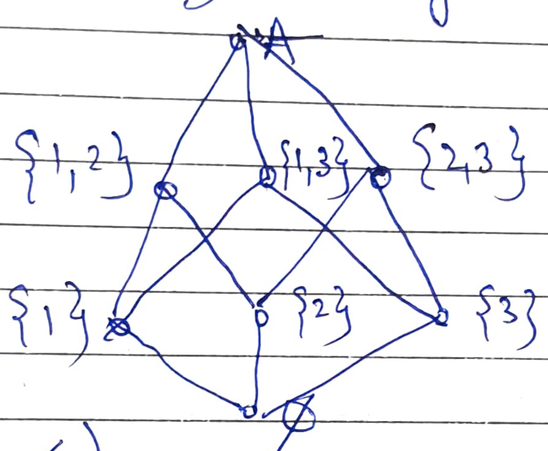
If a & b are non comparable elements of a poset P, we write it as $a \parallel b$.

Defn Any Poset $(P, \leq)$ (Specially finite Posets) can be pictorically represented by HASSE Diagrams.

ex $A = \{1, 2, 3\}$

$(\mathcal{P}(A), \subseteq)$ is a Poset.

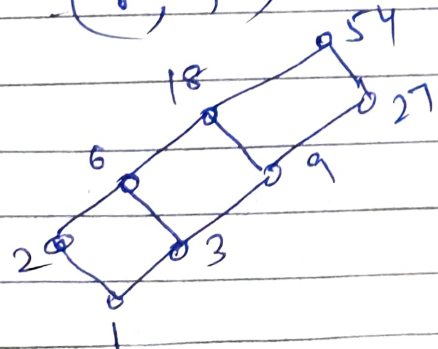
Hasse diagram of $\mathcal{P}(A)$ is:



ex $(\mathbb{N}, \leq)$

ex P = factors of $54 = \{1, 2, 3, 6, 9, 18, 27, 54\}$

$(P, \text{'}|\text{'})$ is a Poset with Hasse Diagra:



Defn
Discrete Order: Let P be a set. Define $a \le b$ in P iff $a = b$. Then $\le$ is a Partial Order relation on P. Such a relation is called Discrete Order.

ex $P = \{1, 2, 3, 4, 5\}$
Define $a \le b$ in P iff $a = b$
(Discrete Order)
$(P, \le)$ is a Poset with Hasse Diagram:

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5.$$

Defn **Antichain**: A poset $(P, \le)$ is called an Antichain if $\le$ is discrete Order.
i.e $a \le b$ in P iff $a = b$
(i.e If $a \ne b$ then $a \nleq b$).

Remark: Antichain is not negation of chain.

\# Every subset of a chain is a chain.
(Prove)

Defn **Inverse Order**: If $(P, \le)$ is a Poset, define an order '$\ge$' on P as $a \ge b$ in P iff $b \le a$.

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"$\geq$" $\equiv$ Inverse of "$\leq$".

Prove: $\geq$ is reflexive, antisymmetric, transitive on P.

$\therefore$ If $(P, \leq)$ is a Poset, then $(P, \geq)$ is also a Poset.

Defn Greatest elt (Or Top element)

Let $(P, \leq)$ be a Poset & $a \in P$.
a is called a greatest element of P if $x \leq a \ \forall x \in P$.

Result: Greatest elt if it exists is Unique.

Pf Let $(P, \leq)$ be a poset
Let a, a' be 2 greatest elements in P.

Since $a \in P$ & a' = greatest
$\therefore \ a \leq a'$ —— (1)

Since $a' \in P$ & a = greatest
$\therefore \ a' \leq a$ —— (2)

By (1) & (2) & antisymmetry,
$a = a'$

Hence, greatest element, if it exists, is Unique.

Defn least element (Or Bottom element)

Let $(P, \leq)$ be a Poset & $b \in P$.
b is called a least element of P if $b \leq x \ \forall x \in P$.

Result: least element, if it exists, is Unique. (Prove)