Taylor's Series

(Taylor's theorem)

Thm:- Suppose that a function f is analytic throughout a disk $|z-z_0|<R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (|z-z_0|<R_0) \quad -(1)$$

where
$$a_n=\frac{f^{(n)}(z_0)}{n!} \qquad n=0,1,2,\dots \quad -(2)$$

Note:- (1) eq. (1) can be written as

$$f(z)=f(z_0)+\frac{f'(z_0)}{1}(z-z_0)+\frac{f''(z_0)(z-z_0)^2}{2!}+\dots$$

$$(\forall z: |z-z_0|<R_0)$$

(2) Any analytic function at a point z_0 must have a Taylor series at z_0.

(3) When $z_0=0$ and assume f is analytic throughout a disk $|z|<R_0$, then

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n \qquad (|z|<R_0)$$

is called a Maclaurin Series.

Proof:- We first prove the theorem by taking $z_0 = 0$.

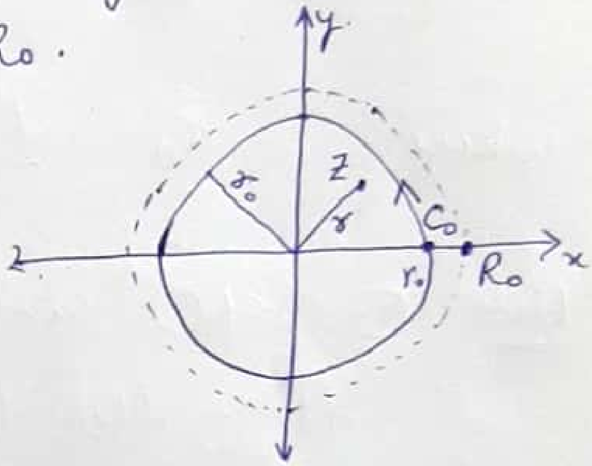
Suppose f is analytic throughout a disk $|z| < R_0$.

Let z be a point such that $|z| < R_0$.

Let $|z| = r$, then $r < R_0$.

Now let C_0 be a positively oriented circle $|z| = r_0$, where $r < r_0 < R_0$.

We will show that f has a power series representation at z.



As f is analytic everywhere inside C_0.

$\therefore$ By Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z}\, ds \quad -(3)$$

Also by extension of Cauchy Integral formula,

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{C_0} \frac{f(s)}{s^{n+1}}\, ds \quad -(4)$$

We have,

$$\frac{1}{s-z} = \frac{1}{s}\left(\frac{1}{1-z/s}\right)$$

$$= \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{z}{s}\right)^k \quad \left(\because \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \text{ where } |z|<1\right)$$

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$$\therefore \quad \frac{1}{s-z} = \sum_{k=0}^{\infty} \frac{z^k}{s^{k+1}}$$

Putting the value in eq. (3), we get

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \sum_{k=0}^{\infty} \frac{f(s) . z^k}{s^{k+1}} ds$$

$$= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \left(\int_{C_0} \frac{f(s) \, ds}{s^{k+1}} \right) . z^k \qquad (\text{why?})$$

Now using eq. (4), we get

$$= \frac{1}{2\pi i} \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(0) . 2\pi i}{k!} \right) . z^k$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} . z^k$$

which is required result, if $z_0 = 0$.
and for all $z : |z| < R_0$.

Now we prove the theorem for arbitrary point z_0.
and suppose that f is analytic when $|z - z_0| < R_0$.

$\therefore$ $f(z)$ is analytic when $|z - z_0| < R_0$.

$\rightarrow$ $f(z + z_0)$ is analytic when $|(z + z_0) - z_0| < R_0$.

$\rightarrow$ $f(z + z_0)$ is analytic when $|z| < R_0$.

Let $g(z) = f(z + z_0)$

$\Rightarrow$ $g(z)$ is analytic when $|z| < R_0$.

$\therefore$ $g(z)$ has a Maclaurin Series expansion.

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^k \qquad |z| < R_0.$$

$$\Rightarrow \quad f(z+z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^k, \quad |z| < R_0$$

Replace z by $z - z_0$, we get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^k, \quad |z - z_0| < R_0.$$

which is desired result. ■.

Examples:-

(1) Let $f(z) = e^z$

then f is an entire function.

(i) around $z = 0$.

$$f^{(n)}(z) = e^z$$

$$\therefore f^{(n)}(0) = 1.$$

$$\therefore e^z = f(0) + f^{(1)}(0).z + \frac{f^{(2)}(0)}{2!} z^2 + \cdots$$

$$= 1 + z + \frac{z^2}{2!} + \cdots$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty.$$

(ii) around $z=1$.

then $e^z = f(1) + f^{(1)}(1)(z-1) + \frac{f^{(2)}(1)(z-1)^2}{2!} + \cdots$

$$= e + e(z-1) + e\frac{(z-1)^2}{2!} + \cdots$$

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}, \quad |z-1| < \infty.$$

(2) find Maclaurin series for $f(z) = \sin z$.

$\to$ As $f(z) = \sin z$ is an entire function.

$\therefore$ By Taylor's theorem, we have

$$\sin z = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, z^n}{n!}.$$

$f^{(1)}(z) = \cos z$, $\quad f^{(2)}(z) = -\sin z$

$f^{(1)}(0) = 1$ $\quad f^{(2)}(0) = 0$

lly $\quad f^{(3)}(z) = -\cos z$, $\quad f^{(4)}(z) = \sin z$

$f^{(3)}(0) = -1$ $\quad f^{(4)}(0) = 0$.

$\therefore \quad f^{(2n)}(0) = 0$

and $f^{(2n+1)}(0) = (-1)^n$, $\quad n = 0, 1, 2, \ldots$

$$\therefore \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad |z| < \infty.$$

Similarly, $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!}$ $(|z|<\infty)$

(3.) find Maclaurin series for $f(z) = \sinh z$.

→ As $\sinh z = \frac{e^z - e^{-z}}{2}$

and $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and $e^{-z} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}$

$\therefore$ $\sinh z = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right]$

$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}\right]$

$= \frac{1}{2}\sum_{n=0}^{\infty} [1-(-1)^n] \frac{z^n}{n!}$

As $1-(-1)^n = 0$, when n is even

& $1-(-1)^n = 2$, when n is odd

$\therefore$

$\sinh z = \frac{1}{2}\sum_{n=0}^{\infty} 2 \cdot \frac{z^{2n+1}}{(2n+1)!}$

$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$, $|z|<\infty$

IIIly $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$, $|z|<\infty$

(4.) find Maclaurin ~~de~~ Series expansion of

$$f(z) = \frac{z}{z^4+9}.$$

→ As $f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+z^4/9}$

$$= \frac{z}{9} \cdot \frac{1}{1-(-z^4/9)}$$

As $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

$$\therefore \frac{1}{1-(-z^4/9)} = \sum_{n=0}^{\infty} \left(-\frac{z^4}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{9^n}.$$

$$\therefore f(z) = \frac{z}{9} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{9^{n+1}}.$$

H.W. Try exercise questions:-.

Laurent Series:-

If a function f fails to be analytic at a point z_0, then we cannot apply Taylor's theorem at that point, but we may still find a series representation for f.

Theorem:- Suppose that f is analytic throughout an annular domain $R_1 < |z-z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in $R_1<|z-z_0|<R_2$. Then, at each point in the domain, $f(z)$ has the series representation.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad (R_1<|z-z_0|<R_2) \quad \text{--- (1)}$$

where
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n=0,1,2,\ldots)$$

and
$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz \qquad (n=0,1,2,\ldots)$$

The series is called Laurent's Series.

[Proof of this theorem is not in syllabus).

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eq. (1) can be written as,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=-\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}} \quad \text{(replacing } n \text{ by } -n)$$

where $b_{-n} = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}$, $(n=-1,-2,\cdots)$

$\therefore$ eq. (1) can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n \quad (R_1 < |z-z_0| < R_2) \quad \text{---} (2)$$

where $C_n = \begin{cases} b_{-n}, & \text{when } n \leq -1 \\ a_n, & \text{when } n \geq 0 \end{cases}$

and $C_n = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}$. $(n=0,\pm 1,\pm 2,\cdots)$

Examples:-

(1). Laurent Series of $f(z) = e^{1/z}$.

$\to$ As $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots \quad (|z| < \infty)$

Replacing z by $1/z$, we get,

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots \quad (0 < |z| < \infty). \quad \text{---} (3)$$

is the Laurent Series representation.

Comparing eq. ③ with eq. ①, we get

$$b_1 = 1 .$$

and $b_1 = \frac{1}{2\pi i} \int_C e^{1/z} \, dz$, where C is positively oriented closed contour around origin.

$$\therefore \frac{1}{2\pi i} \int_C e^{1/z} \, dz = 1$$

$$\Rightarrow \boxed{\int_C e^{1/z} \, dz = 2\pi i .}$$

② Let $f(z) = \frac{-1}{(z-1)(z-2)}$

Derive series representation of f in powers of z.

$\rightarrow$ $f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$

$\rightarrow$ f has two singular points $z=1$ and $z=2$ and f is analytic in the domain,

$D_1 := |z| < 1$, $D_2 := 1 < |z| < 2$, $D_3 := 2 < |z| < \infty$.

We will show f has series representation in Domains D_1, D_2 and D_3 .

(i) <u>In Domain D_1</u>,

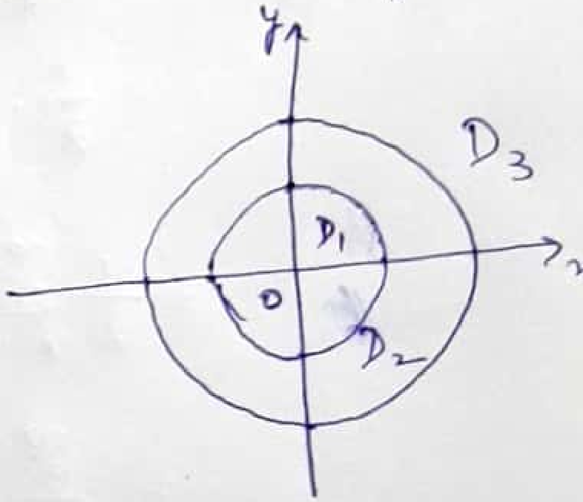
let $z \in D_1$ then $|z| < 1 \Rightarrow |z/2| < 1$

then $f(z) = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{(1-z/2)}$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$f(z) = \sum_{n=0}^{\infty}(2^{-n-1}-1)z^n \quad , \quad (|z|<1)$$

This representation is Maclaurin Series representation



(ii) <u>In Domain D_2</u>

let $z \in D_2$

then $1 < |z| < 2$

$\Rightarrow |1/z| < 1$ and $|z/2| < 1$.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z}\cdot\frac{1}{1-1/z} + \frac{1}{2}\cdot\frac{1}{1-z/2}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \quad . \quad (1<|z|<2)$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \quad , \quad (1<|z|<2).$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad (1<|z|<2)$$

This is laurent series of f in Domain D_2.

(iii) <u>In Domain D_3.</u>

Let $z \in D_3$.

then $2 < |z| < \infty$

$\Rightarrow$ $|2/z| < 1$ and $|1/z| < 1$.

As $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$

$$= \frac{1}{z} \cdot \frac{1}{1-1/z} - \frac{1}{z} \cdot \frac{1}{1-2/z}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} - \frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \quad (2<|z|<\infty)$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \quad (2<|z|<\infty)$$

This is laurent series representation where all a_n's are zero.