Recurrence Formulae for $J_n(x)$

(I) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Sol: We know that $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \quad \text{---(1)}$$

Diff (1) w.r.t x

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \,(n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

or $$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (n+2r) \left(\frac{x}{2}\right) \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \, n}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot 2r \left(\frac{x}{2}\right) \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r \, x}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

put $r-1=s$

$$= n J_n(x) + \sum_{s=0}^{\infty} \frac{(-1)^{s+1} \, x}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= n J_n(x) - \sum_{s=0}^{\infty} \frac{(-1)^{s} \, x}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$\boxed{x J_n'(x) = n J_n(x) - x J_{n+1}(x)}$$

(II) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

Sol:- We know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Differentiate above w.r.t x

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

Multiplying above eqn by x.

$$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x(2n-n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} \quad ***$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{x} (-n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{x}{2}$$

$$= -n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)}\, x \left(\frac{x}{2}\right)^{n-1+2r}$$

$$= -n J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n-1+2r}$$

$$\therefore \boxed{x J_n'(x) = -n J_n(x) + x J_{n-1}(x)} \longrightarrow \text{(II)}$$

$***$ $\quad x\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2} = \cancel{x}\left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\frac{x}{2}} \cdot \frac{1}{\cancel{x}}$

$$= \left(\frac{x}{2}\right)^{n+2r}$$

(III) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

Sol. We have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Differentiate above eqn w.r.t x and multiply both side by 2

or $$2J_n'(x) = 2\sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}\cdot\frac{1}{2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$\cancel{= \sum_{r=0}^{\infty} \frac{(-1)^r\,(2n+2r-n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}}$$

$$\cancel{= \sum_{r=0}^{\infty} \frac{(-1)^r\,2(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r\,(-n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+r+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r\,r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= J_{n-1}(x) + \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+1+r+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$ [Put $r-1=s$

$$= J_{n-1}(x) - \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$\boxed{2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)} \qquad \text{—— (III)}$$

(IV) $2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Sol: $J_n(x) = \sum\limits_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Multiply both side of above eqn by $2n$

$$2n\,J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2n}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

add & subtract $2r$ with $2n$ of numerator of r.h.s

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{x}{2}$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^r (-2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{x}{2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, x}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r (-x)}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, x}{r!\,\Gamma(n+r-1+1)} \left(\frac{x}{2}\right)^{n-1+2r} + x\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$= x\,J_{n-1}(x) + x\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s} \qquad [\text{st. } r-1=s$$

$$= x\,J_{n-1}(x) + x\,J_{n+1}(x)$$

$$2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Or

$$\boxed{\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)} \longleftarrow \text{(IV)}$$

2002

$$\frac{2(n+1)}{x} J_{n+1}(x) = J_n(x) + J_{2n+1}(x)$$

If we st. $n = n+1$.

(V) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Sol:- $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -n x^{-n-1} J_n(x) + x^{-n} J_n'(x)$

$= -n x^{-n-1} J_n(x) + x^{-n-1+1} J_n'(x)$

$= x^{-n-1}\left[-n J_n(x) + x J_n'(x)\right]$ ---- (1)

from (I) relation $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

st. in eqn (1) we get

$= x^{-n-1}\left[\cancel{-n J_n(x) + n J_n(x)} - x J_{n+1}(x)\right]$

$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n-1} \cdot (-x) J_{n+1}(x)$

$$\boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)} \quad \text{——— (V)}$$

(VI) $\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n} J_{n-1}(x)$

Sol: $\frac{d}{dx}\left[x^{n} J_n(x)\right] = n x^{n-1} J_n(x) + x^{n} J_n'(x)$

$= n x^{n-1} J_n(x) + x^{n-1+1} J_n'(x)$

$= x^{n-1}\left[n J_n(x) + x J_n'(x)\right]$ ---- (1)

from (II) relation $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

st. in (1) $= x^{n-1}\left[\cancel{n J_n(x)} - \cancel{n J_n(x)} + x J_{n-1}(x)\right]$

$\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n-1} \cdot x J_{n-1}(x)$

$$\boxed{\frac{d}{dx}\left[x^{n} J_n(x)\right] = x^{n} J_{n-1}(x)} \quad \text{——— (VI)}$$

Generating Function for $J_n(x)$

$$\boxed{e^{x(z-\frac{1}{z})/2} = \sum_{n=-\infty}^{+\infty} z^n J_n(x)}$$

Orthonormality of Bessel's Function

★★★ 2000 | 2002 | 2009

Show that $\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$ for $\alpha \neq \beta$

Sol. Bessel's eqn is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \quad \text{------} (1)$$

Consider two Bessel's function of first kind of order n

$$u = J_n(\alpha x) \quad \& \quad v = J_n(\beta x) \quad \text{--------} (2)$$

st. αx for x and u for y in (1) we get

$$\frac{d^2u}{d(\alpha x)^2} + \frac{1}{\alpha x}\frac{du}{d(\alpha x)} + \left(1 - \frac{n^2}{(\alpha x)^2}\right)u = 0$$

or $$\frac{1}{\alpha^2}\frac{d^2u}{dx^2} + \frac{1}{\alpha^2 x}\frac{du}{dx} + \left(1 - \frac{n^2}{\alpha^2 x^2}\right)u = 0$$

or $$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2 x^2 - n^2)u = 0 \quad \text{-----} (3)$$

Similarly if we st. βx for x & v for y then from (1) we get

$$x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} + (\beta^2 x^2 - n^2)v = 0 \quad \text{--------} (4)$$

multiply ③ by $\frac{v}{x}$ & ④ by $\frac{u}{x}$ & subtracting we get

$$\frac{v}{x}\left[x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+(\alpha^2x^2-n^2)u\right]-\frac{u}{x}\left[x^2\frac{d^2v}{dx^2}+x\frac{dv}{dx}+(\beta^2x^2-n^2)v\right]=0$$

$$x\left[v\frac{d^2u}{dx^2}-u\frac{d^2v}{dx^2}\right]+\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)+(\alpha^2-\beta^2)xuv=0$$

or $$\frac{d}{dx}\left[x\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]+(\alpha^2-\beta^2)xuv=0$$

using ② we get

$$\frac{d}{dx}\left[x\left\{J_n(\beta x)\frac{d}{dx}J_n(\alpha x)-J_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right\}\right]$$

$$+(\alpha^2-\beta^2)xJ_n(\alpha x)J_n(\beta x)=0 \quad ----- ⑤$$

integrating ⑤ w.r.t x within limit 0 to 1

$$\left\{xJ_n(\beta x)\frac{d}{dx}J_n(\alpha x)-xJ_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right\}_0^1$$

$$\int_0^1(\alpha^2-\beta^2)xJ_n(\alpha x)J_n(\beta x)\,dx=0 \quad ----- ⑥$$

If α & β are different, then $J_n(\alpha)=J_n(\beta)=0$ are all finite

$\therefore$ $(\alpha^2-\beta^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx=0$

as $\alpha\neq\beta$

$\therefore$ $$\boxed{\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx=0}$$

$$*\ \frac{v}{x}x^2\frac{d^2u}{dx^2}+\frac{v}{x}x\frac{du}{dx}+\frac{\alpha^2vx^2u}{x}-\frac{n^2uv}{x}-\frac{u}{x}x^2\frac{d^2v}{dx^2}-\frac{u}{x}x\frac{dv}{dx}$$

$$-\frac{\beta^2x^2uv}{x}+\frac{un^2v}{x}=0$$

Hermite Differential Equation

$$\boxed{\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0}$$

where 'n' is a constt.

Its complete solⁿ. is given by

$$\boxed{y = Ay_1 + By_2}$$ where A & B are arbitrary constts. &

y_1 & y_2 are given by (solⁿs.)

$$y_1 = a_0\left[1 - \frac{2n}{2!}x^2 + 2^2\frac{n(n-2)}{4!}x^4 + \cdots + \frac{(-2)^m n(n-2)\cdots(n-2m+2)}{2m!}x^{2m} + \cdots\right]$$

&

$$y_2 = a_0\left[x - \frac{2(n-1)}{3!}x^3 + \frac{2^2(n-1)(n-3)}{5!}x^5 + \cdots + (-2)^m\frac{(n-1)(n-3)\cdots(n-2m+1)}{(2m+1)!}x^{2m+1} + \cdots\right]$$

Hermite Polynomial

Hermite Polynomial of degree 'n', for n being a +ve integer represented by

$$\boxed{H_n(x) = \sum_{r=0}^{p} (-1)^r \frac{n!}{r!\,(n-2r)!}(2x)^{n-2r}}$$

where $p = \begin{cases} n/2 & \text{for } n = \text{EVEN} \\ \frac{n-1}{2} & \text{for } n = \text{ODD} \end{cases}$

n	$H_n(x)$	Explanation
0	$H_0(x)=1$	$\sum_{r=0}^{0} \frac{(-1)^r\, 0!}{r!\,(-2r)!}(2x)^{-2r} = \frac{(-1)^0 \cdot 1}{1\cdot 1}(2x)^0 = 1$
1	$H_1(x)=2x$	$\sum_{r=0}^{0} \frac{(-1)^r\, 1!}{0!\,(1-2r)!}(2x)^{1-2r} = \frac{(-1)^0\, 1!}{1\cdot 1!}(2x)\cdot(2x)^{-2\cdot 0}$ $= 2x$
2	$H_2(x)=4x^2-2$	$\sum_{r=0}^{1} \frac{(-1)^r\, 2!}{r!\,(2-2r)!}(2x)^{2-2r}$ $= \frac{(-1)^0\, 2!}{0!\,(2-0)!}(2x)^2\cdot(2x)^{-2\times 0} + \frac{(-1)^1\, 2!}{1!\,(2-2)!}(2x)^0$ $= \frac{2}{2!}4x^2 - 2$ $= 4x^2-2$

(4)

like it.

$$\Rightarrow H_3(x) = \sum_{r=0}^{1} \frac{(-1)^r \, 3!}{r!\,(3-2r)!} (2x)^{3-2r}$$

$$= \frac{(-1)^0 \cdot 3!}{0!\,3!} (2x)^3 + \frac{(-1)^1 \cdot 3!}{1!\,1!} (2x)^1$$

$$= 8x^3 - 12x$$

$$\Rightarrow H_4(x) = \sum_{r=0}^{2} \frac{(-1)^r \, 4!}{r!\,(4-2r)!} (2x)^{4-2r}$$

$$= \frac{(-1)^0 \cdot 4!}{0!\,4!} (2x)^4 + \frac{(-1)^1 \cdot 4!}{1!\,2!} (2x)^2$$

$$+ \frac{(-1)^2 \, 4!}{2!\,0!} (2x)^0$$

$$= 16x^4 - 48x^2 + 12$$

& . so on

Generating function of a Hermite Polynomial

$$\boxed{f(x,z) = e^{2zx - z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n}$$

is called Generating function of Hermite Polynomial.

Proof:- we have $e^{2zx - z^2} = e^{2zx} \cdot e^{-z^2}$

$$= \sum_{r=0}^{\infty} \frac{(2zx)^r}{r!} \cdot \sum_{s=0}^{\infty} \frac{(-z^2)^s}{s!}$$

$$= \sum_{r,s=0}^{\infty} \frac{(-1)^s (2x)^r}{r!\, s!} \cdot z^{r+2s} \quad \to n$$

on RHS the coeff. of z^n is obtained by putting $n = r+2s$ (for fixed value of s) ie $r = n-2s$

is given by $\dfrac{(-1)^s (2x)^{n-2s}}{(n-2s)!\, s!}$

The total coeff. of z^n is obtained by summing over all allowed values of s and since $r = n-2s$

$\therefore$ $n-2s \geq 0$ or $s \leq \frac{n}{2}$

thus if $n = \text{EVEN}$ s tends to 0 to $\frac{n}{2}$

if $n = \text{ODD}$ s tends to 0 to $\frac{n-1}{2}$

$$\left[\sum_{s=0}^{n/2} \quad \sum_{s=0}^{n-1/2} \right]$$

$$\therefore \text{coeff. of } z^n = \sum_{s=0}^{n/2} \frac{(-1)^s (2x)^{n-2s}}{(n-2s)!\, s!} = \frac{H_n(x)}{n!}$$

Hence we may write :-

$$\boxed{e^{2zx - z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n}$$

Note:- $e^{2zx - z^2} = e^{x^2 - (z-x)^2}$ generates all Hermite Polynomial is known as Generating function.

(6) ## Rodrigues formula for Hermite Polynomial

$$\boxed{H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)}$$

Proof:- Generating function is given by

$$e^{2zx-z^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}z^n$$

$$e^{x^2-(z-x)^2} = \frac{H_0(x)}{0!}z^0 + \frac{H_1(x)}{1!}z + \frac{H_2(x)}{2!}z^2 + \dots$$

$$+ \frac{H_n(x)}{n!}z^n + \frac{H_{n+1}(x)}{(n+1)!}z^{n+1} + \dots$$

differentiating b.s. w.r.t 'z', 'n' times partially then put $z=0$ we get

$$\left[\frac{d^n}{dz^n}\cdot e^{x^2-(z-x)^2}\right]_{z=0} = \frac{H_0(x)}{0!}\times 0 + \frac{H_1(x)}{1!}\cdot 1 + \frac{H_2(x)}{2!}2\cdot z + \dots$$

$$+ \frac{H_3(x)}{3!}3\cdot z^2 + \dots$$

$$\frac{1}{1}\times 0 + 2x + \frac{(4x^2-2)}{2!}2\cdot z$$

$$= \frac{H_n(x)}{n!}n!$$

or $$H_n(x) = e^{x^2}\cdot\left[\frac{d^n}{dz^n}e^{-(z-x)^2}\right]_{z=0}$$

st. $z-x=t$ ie at $z=0$, $-x=t$

$\therefore$

Recurrance formulae

I) $\boxed{H'_n(x) = 2n H_{n-1}(x)}$

Proof: Generating function is given by

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2zx - z^2} \quad \text{--- (1)}$$

Diff. (1) w.r.t. 'x' we get

$$\sum_{n=0}^{\infty} \frac{H'_n(x)}{n!} z^n = e^{2zx - z^2} \cdot 2z$$

$$= 2z \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

$$= 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^{n+1}$$

$$= 2 \sum_{n=0}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} z^n \qquad \left[\begin{array}{l} \text{as } n+1 \to n \\ n \to n-1 \end{array}\right]$$

equating coeff of z^n on b.s.

$$\frac{H'_n(x)}{n!} = 2 \frac{H_{n-1}(x)}{(n-1)!}$$

$$H'_n(x) = \frac{2 H_{n-1}(x)}{(n-1)!} \cdot n!$$

$$= \frac{2 H_{n-1}(x) \cdot n (n-1)!}{(n-1)!}$$

$$\Rightarrow \boxed{H'_n(x) = 2n H_{n-1}(x)}$$

(8)

II $\boxed{2x H_n(x) = 2n H_{n-1}(x) + H_{n+1}(x)}$

Proof:- $$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2zx - z^2} \quad \text{--- (1)}$$

Diff. (1) w.r.t z, we get

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} n z^{n-1} = e^{2zx-z^2} (2x - 2z)$$

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{\cancel{n}(n-1)!} \cancel{n} \cdot z^{n-1} = 2x e^{2zx-z^2} - 2z e^{2zx-x^2}$$

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} z^{n-1} = 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n - 2z \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

$$\sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} z^{n-1} = 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n - 2z \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

$\left(\text{as } \sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} z^{n-1} = 0 \text{ as } n = 0 \right.$

$$\therefore 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = 2z \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n + \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} z^{n-1}$$

equating the coeffs. of z^n on b.s., we get

$$2x \frac{H_n(x)}{n!} = \frac{2 H_{n-1}(x)}{(n-1)!} + \frac{H_{n+1}(x)}{n!}$$

$\left[\text{as } n! = n(n-1)! \right.$

$$\frac{2x H_n(x)}{n} = 2 H_{n-1}(x) + \frac{H_{n+1}(x)}{n}$$

$\left(\text{as } -n! = \infty \ \& \ \frac{1}{\infty} = 0 \right)$

$$\Rightarrow \boxed{2x H_n(x) = 2n H_{n-1}(x) + H_{n+1}(x)}$$

III $\boxed{H_n'(x) = 2x H_n(x) - H_{n+1}(x)}$

Proof:- $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2zx - z^2}$ — (1)

rel^n (I) is $H_n'(x) = 2n H_n(x) - H_{n+1}(x)$ — (2)

(II) is $2x H_n(x) = 2n H_{n-1}(x) + H_{n+1}(x)$ — (3)

Now (2) − (3), we get

$H_n'(x) - 2x H_n(x) = \cancel{2n H_n(x)} - H_{n+1}(x) - \cancel{2n H_{n-1}} - H_{n+1}$

$\therefore \boxed{H_n'(x) = 2x H_n(x) - H_{n+1}(x)}$ Hence Proved

IV $\boxed{H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0}$

Proof:- Hermite Diff is $y'' - 2xy' + 2ny = 0$

as $H_n(x)$ is the sol^n of Hermite Diff. Eqn.

so substituting $H_n(x)$ for y, we get

$$\boxed{H_n''(x) - 2n H_n'(x) + 2n H_n(x) = 0}$$