

Momentum Eigenfunction:

$$\hat{p} e^{ikx} = \hbar k e^{ikx}$$

$$\hat{p} e^{-ikx} = -\hbar k e^{-ikx}$$

So, e^{ikx} & e^{-ikx} are eigenfunction of momentum operator with eigenvalues $\hbar k$ & $-\hbar k$. These functions are also momentum eigenfunction.

Since $\psi(x,t) = A e^{i(kx - \frac{Et}{\hbar})}$

So, $|\psi(x,t)| = |A|^2$ The position probability density is independent of time as well as position x . Thus the position of a particle rep. by plane wave is completely unknown. This is in accordance with the uncertainty relation because if a particle has $\Delta p = 0$ then $\Delta x = \infty$. So a plane wave may be physically interpreted as particle of well defined momentum moving in a beam of unlimited length. The probability current-density

$$\begin{aligned} j &= \text{Re} \left[\psi^* \frac{\hbar}{im} \frac{\partial \psi}{\partial x} \right] \\ &= \text{Re} \left[A^* e^{-ikx} \left(\frac{\hbar}{im} \right) (ik) A e^{ikx} \right] \\ &= \frac{\hbar k}{m} |A|^2 = \frac{p}{m} |A|^2 = v |A|^2 \end{aligned}$$

j is also independent of x & t as expected for stationary states.

Stationary states of free particle are propagating waves & their wavelength is

$$\lambda = \frac{2\pi}{k} \quad \& \quad p = \hbar k \quad E = \frac{\hbar^2 k^2}{2m}$$

So, $V_{\text{quantum/wave}} = \frac{\omega}{k} = \frac{\hbar \omega}{\hbar k} = \frac{p^2}{2m p} = \frac{p}{2m} = \frac{\hbar k}{2m}$

$$V_{\text{classical}} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2}{m} \frac{p^2}{2}} = \frac{p}{m} = \hbar k = 2 V_{\text{quantum}}$$

Hence the particle travels twice as far τ
as the wave that represents it. So,

→ Also, The wave function for free particle is not normalizable i.e.

$$\int_{-\infty}^{+\infty} \psi^*(x) \psi(x) dx = |A|^2 \int_{-\infty}^{+\infty} dx$$

$$= |A|^2 \infty$$

→ Hence the solution are unphysical F
because physical wave function
must be square integrable. So,

→ So a free particle cannot exist in a
stationary state or the free particle
do not have definite energy states.

→ Hence plane waves cannot be the physi.
acceptable solution of free particle S.E.

→ We can construct physical solution by
means of linear superposition of
plane wave which is known as wave
packets.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$$

$\phi(k)$ = Amplitude of wave packet.

To determine $\phi(k)$ we consider

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} dk \quad \text{--- (2)}$$

This is the Fourier analysis problem
So to find $\phi(k)$ we use Parseval
theorem i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k) e^{ikx} dk$$

$$\Leftrightarrow F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx.$$

$F(k)$ is called Fourier transform of
 $f(x)$

So, the solution of generic quantum
problem

$$\boxed{\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x, 0) e^{-ikx} dx.} \quad (3)$$

A free particle initially localised
in range $-a < x < a$ is released at
time $t = 0$

$$\psi(x, 0) = \begin{cases} A & \text{if } -a < x < a \\ 0 & \text{otherwise.} \end{cases}$$

Find $\phi(k)$

First normalise.

$$\int_{-a}^a |\psi(x, 0)|^2 dx = |A|^2 \int_{-a}^a dx = 2a|A|^2 = 1$$

$$\Rightarrow A = \frac{1}{\sqrt{2a}}$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a}} \int_{-a}^{+a} e^{-ikx} dx = \frac{1}{2\sqrt{\pi a}} \left. \frac{e^{-ikx}}{-ik} \right|_{-a}^a$$

$$\boxed{\phi(k) = \frac{1}{\sqrt{\pi}} \frac{\sin ka}{k}}$$

A localised wavefunction is called as a wavepacket. A wavepacket is more like a pulse which has appreciable amplitude only in localised space & time. This means that for a certain position & time the amplitude is considerable whereas for rest of the space & time domain it is almost zero. These kinds of packets can be considered/constructed with superposition of many plane waves of different nearby λ & ν . So these packets will contain a band of wavelengths & frequencies.

From eqn (2) it can be seen that $|\psi(x,0)|$ peaks at $x=0$ & vanishes away from $x=0$. Similarly $\phi(k)$ peaks at $k=0$.

$\{\psi(x,0)\} \equiv$ wavefunction or prob. ^{amp.} of finding the particle at position x . Hence $|\psi(x,0)|^2 \rightarrow$ Probability density for finding particle at x & $p(x)dx = \text{prob.}$ " b/w x & $x+dx$.

$$\text{As, } \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{+\infty} |\phi(k)|^2 dk$$

Thus if $\psi(x)$ is normalised so is $\phi(k)$ & vice versa. Hence $\phi(k)$ can be interpreted as probability amplitude for measuring a wave packet for particle in the state $\phi(k)$.

WAVE PACKET : The simple plane wave does not fit to be a good wavefunction as it is not bounded. Also for plane wave phase velocity $> c$ that is not possible. So also plane wave represents a particle having definite momentum so it lacks ~~complete~~ localisation of particle in space. It is represented by

$$\psi(x,t) = A e^{i(kx - \omega t)}$$

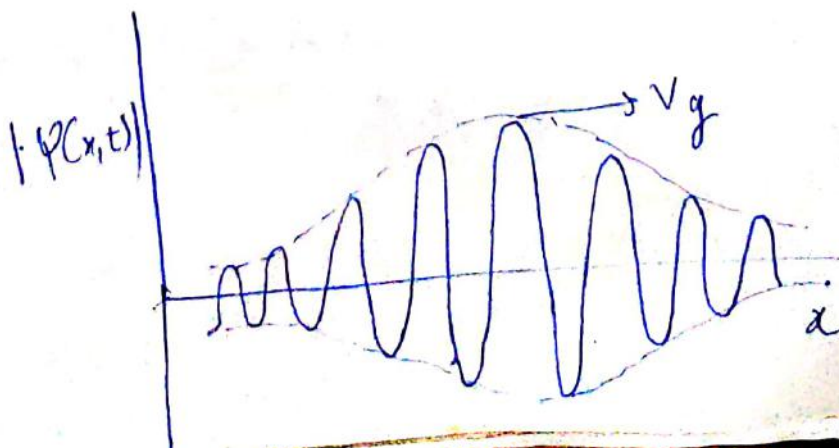
$|\psi(x,t)|^2 = A^2$ i.e. independent of position. So particle has equal probability of being found $-\infty$ to $+\infty$. So a plane wave cannot represent a localised particle.

So we form a wave packet to represent a particle. A wave packet can be formed by superposing plane waves of different wave numbers in such a way that they interfere destructively outside of a given region of space. Then the wave packet is formed by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega t)} dk$$

Amplitude function

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{-i(kx - \omega t)} dx.$$



* A particle is characterized by its mass & energy
& as wave is characterized by its amplitude
phase, wave vector.

* Wave is a pattern that travels & propagates
energy.

* Wave exhibit particle behaviour at microscopic
level.

→ 1 - D T DSE.

→ Conditions to be satisfied by S.E.

→ 1) Acceptable wave function.

→ 2) Postulates of Quantum Mechanics.

→ 3)

$$\frac{d^2 y}{dx^2} + ay = 0.$$

(18)

$|\phi(k)|^2 =$ Prob. density for measuring k as particles wave vector.
 $P(k)dk = |\phi(k)|^2 dk$ probability of finding the particles wave vector b/w k & $k+dk$.

Exact Statement + proof of Uncertainty Relation. Position-Mom.

Uncertainty is defined as the root mean square deviation from mean value (or expectation value)

$$\Delta x = \langle (x - \langle x \rangle)^2 \rangle^{1/2} \quad \text{--- (1)}$$

$$\Delta p = \langle (p - \langle p \rangle)^2 \rangle^{1/2} \quad \text{--- (2)}$$

Let $A = x - \langle x \rangle$ & $B = p - \langle p \rangle$

$$\Rightarrow B = \frac{1}{i\hbar} \left[\frac{d}{dx} - \left\langle \frac{d}{dx} \right\rangle \right]$$

Then

$$\begin{aligned} (\Delta x)^2 (\Delta p)^2 &= \int_{-\infty}^{+\infty} \psi^* A^2 \psi dx \int_{-\infty}^{+\infty} \psi^* B^2 \psi dx \\ &= \int_{-\infty}^{+\infty} (A^* \psi^*) (A \psi) dx \int_{-\infty}^{+\infty} (B^* \psi^*) (B \psi) dx \end{aligned} \quad \text{--- (3)}$$

Since A & B are Hermitian operator. So we can write as above. (4)

Using Schwarz inequality

$$\int |f|^2 dx \int |g|^2 dx \geq \left| \int f^* g dx \right|^2 \quad \text{--- (5)}$$

Where f & g are arbitrary function & equality is valid only if $f = \alpha g$; $\alpha = \text{constant}$.

Taking $f = A\psi$ & $g = B\psi$, eqn (5) becomes.

$$\begin{aligned}
 (\Delta x)^2 (\Delta p)^2 &\geq \left| \int (A^* \psi^*) (B\psi) dx \right|^2 \\
 &= \left| \int (\psi^* AB\psi) dx \right|^2 \\
 \left| \int \psi^* AB\psi \right|^2 &= \left| \int \psi^* \left[\frac{1}{2}(AB-BA) + \frac{1}{2}(AB+BA) \right] \psi dx \right|^2 \\
 &= \frac{1}{4} \left| \int \psi^* (AB-BA) \psi dx \right|^2 \\
 &\quad + \frac{1}{4} \left| \int \psi^* (AB+BA) \psi dx \right|^2
 \end{aligned}$$

$$\begin{aligned}
 (AB-BA)\psi &= -i\hbar \left[x \frac{d\psi}{dx} - \frac{d}{dx}(x\psi) \right] \\
 &= -i\hbar \left[x \frac{d\psi}{dx} - \psi - x \frac{d\psi}{dx} \right] \\
 &= i\hbar \psi
 \end{aligned}$$

$$\text{So, } \int \psi^* (AB-BA) \psi dx = i\hbar \int \underbrace{\psi^* \psi}_{=1} dx$$

$$\text{So, } (\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4} |i\hbar|^2 \geq \frac{\hbar^2}{4}$$

$$\text{or } \boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$$

Find $\psi(x,0)$ for a gaussian wave packet
 $\phi(k) = A \exp\left[-\frac{a^2}{4} (k-k_0)^2\right]$ where A
is normalisation constant

$$\int_{-\infty}^{+\infty} |\phi(k)|^2 dk = 1$$

$$\Rightarrow |A|^2 \int_{-\infty}^{+\infty} \exp\left(-\frac{a^2}{4} (k-k_0)^2\right) dk = 1$$

$$\text{Let } z = k - k_0 \quad dz = dk$$

$$\Rightarrow |A|^2 \int_{-\infty}^{+\infty} \exp\left(-\frac{a^2 z^2}{4}\right) dz = 1$$

$$\text{Let } \frac{a^2 z^2}{4} = p \quad \therefore \frac{a^2}{2} z dz = dp$$

$$\Rightarrow dz = \frac{dp}{a^2 z} = \frac{dp}{a^2 \sqrt{2p}}$$

$$\therefore |A|^2 \int_{-\infty}^{+\infty} e^{-p} \frac{1}{\sqrt{2p}} p^{-1/2} dp = 1$$

$$\Rightarrow \frac{|A|^2}{\sqrt{2} a} \times 2 \int_0^{\infty} e^{-p} p^{-1/2} dp = 1$$

$$\Rightarrow \frac{\sqrt{2}}{a} |A|^2 \Gamma\left(\frac{1}{2}\right) = 1 \quad \left[\because \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \right]$$

$$\Rightarrow |A|^2 = \frac{a}{\sqrt{2\pi}}$$

$$\phi(k) = \left(\frac{a^2}{2\pi}\right)^{1/4} \exp\left(-\frac{a^2}{4} (k-k_0)^2\right)$$

$$\therefore \psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{a^2}{2\pi}\right)^{1/4} e^{-\frac{a^2}{4}(k-k_0)^2} e^{i'(k-k_0)x} e^{ik_0x} dx$$

$$\text{Let } k' = k - k_0 \quad dk' = dk$$

$$\therefore \psi_0(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{a^2}{2\pi}\right)^{1/4} e^{ik_0x} \int_{-\infty}^{+\infty} e^{-\frac{a^2 k'^2}{4}} e^{ik'x} dk'$$

$$-\frac{a^2 k'^2}{4} + ik'x = -\left[\frac{a^2 k'^2}{4} - ik'x\right]$$

$$= -\left[\left(\frac{ak'}{2}\right)^2 - 2\frac{ak'}{2} \frac{ix}{a} + \left(\frac{ix}{a}\right)^2 + \frac{x^2}{a^2}\right]$$

$$= -\left[\left(\frac{ak'}{2} - \frac{ix}{a}\right)^2 + \frac{x^2}{a^2}\right]$$

$$\text{Let } y = \frac{ak'}{2} - \frac{ix}{a}$$

$$dy = \frac{a}{2} dk'$$

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{a^2}{2\pi}\right)^{1/4} e^{ik_0x} \frac{2}{a} e^{-x^2/a^2} \int_{-\infty}^{+\infty} e^{-y^2} dy$$

$$\psi_0(x) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{ik_0x} e^{-x^2/a^2}$$

Gaussian wavepacket & its spread with time (20)

The function

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx - \omega t)} dk$$

then $\phi(k)$ is gaussian function

$$\phi(k) = \left(\frac{a^2}{2\pi}\right)^{1/4} e^{-\frac{a}{4}(k-k_0)^2}$$

$$\therefore \psi(x,t) = \frac{1}{\sqrt{2\pi}} \left(\frac{a^2}{2\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-\frac{a}{4}(k-k_0)^2} e^{i(kx - \omega t)} dk$$

At $t=0$

$$\psi(x,0) = \int_{-\infty}^{+\infty} e^{-\frac{a}{4}(k-k_0)^2} e^{ikx} dk \quad (1)$$

Put $k' = k - k_0$

$$\psi(x,0) = \int_{-\infty}^{+\infty} e^{-ak'^2} e^{i(k'+k_0)x} dk'$$

$$= e^{ik_0x} \int_{-\infty}^{+\infty} e^{-ak'^2} e^{ik'x} dk' \quad (2)$$

$$-ak'^2 + ik'x = -\left[ak'^2 - ik'x\right]$$

$$= -a\left[k'^2 - \frac{ik'x}{a}\right]$$

$$= -a\left[\left(k'\right)^2 - 2 \cdot k' \cdot \frac{ix}{2a} + \left(\frac{ix}{2a}\right)^2 - \left(\frac{ix}{2a}\right)^2\right]$$

$$= -a\left[\left(k' - \frac{ix}{2a}\right)^2 - \frac{i^2x^2}{4a^2}\right]$$

$$= -a\left(k' - \frac{ix}{2a}\right)^2 - \frac{x^2}{4a}$$

Putting this value in eqn (1)

* Spreading or nonspreading of packet is dictated by the form of function $\omega(k)$

$$\begin{aligned}\psi(x,0) &= e^{ik_0 x} \int_{-\infty}^{+\infty} e^{-a(k' - \frac{ix}{a})^2} e^{-\frac{x^2}{4a}} dx \\ &= e^{ik_0 x} e^{-x^2/4a} \int_{-\infty}^{+\infty} e^{-a(k' - \frac{ix}{a})^2} dx\end{aligned}$$

$$\text{Let } k' - \frac{ix}{a} = q$$

$$\therefore dk' = dq$$

$$\psi(x,0) = e^{ik_0 x} e^{-x^2/4a} \int_{-\infty}^{+\infty} e^{-aq^2} dq$$

$$\boxed{\psi(x,0) = B(x) e^{ik_0 x}} \quad (1) \quad \sqrt{\frac{\pi}{a}}$$

$B(x)$ = Amp. of wavepacket
 $e^{ik_0 x} \rightarrow$ phase factor.

$$|\psi(x,0)|^2 = B^2 = \frac{\pi}{a} e^{-x^2/2a} \quad (2)$$

$$\psi(x,t) = \int_{-\infty}^{+\infty} e^{-a(k-k_0)^2} e^{i(kx - \omega t)} dk$$

If $\omega(k) = ck$, ω is linear in k

$$\psi(x,t) = \int_{-\infty}^{+\infty} e^{-a(k-k_0)^2} e^{ik(x-ct)} dk$$

$$\text{So, } \therefore \boxed{\psi(x,t) = \psi(x-ct, 0)}$$

Eqn (2) is similar to eqn (1) except that $x \rightarrow x-ct$

(21)

$$|\psi(x-ct, 0)|^2 = \frac{\hbar}{a} e^{-\frac{(x-ct)^2}{2a}}$$

Thus wavefun $\psi(x, t) = \psi(x-ct, 0)$
 have same shape as that of
 standing wave funⁿ $\psi(x, 0)$
 Thus there is no spreading in time.

But for matter waves, ω is
 not linearly $\propto$ to k

$$\begin{aligned}\omega(k) &= \omega(k_0 + k - k_0) \\ &= \omega(k_0) + (k - k_0) \frac{\partial \omega}{\partial k} \bigg|_{k_0} + \frac{1}{2} (k - k_0)^2 \frac{d^2 \omega}{dk^2} \bigg|_{k_0} + \dots \\ &= \omega(k_0) + (k - k_0) v_g + \frac{1}{2} (k - k_0)^2 \beta + \dots\end{aligned}$$

$$\psi(x, t) = \int_{-\infty}^{+\infty} e^{-a(k-k_0)^2} e^{i[kx - \omega(k_0)t - (k-k_0)v_g t - \frac{1}{2}\beta(k-k_0)^2 t]} dk$$

$$k - k_0 = k'$$

$$\Rightarrow k = k' + k_0$$

$$\begin{aligned}\psi(x, t) &= \int_{-\infty}^{+\infty} e^{-ak'^2} e^{i[k_0 x + k'x]} e^{-i\omega(k_0)t} e^{-ik'v_g t} e^{-i\frac{1}{2}\beta k'^2 t} dk' \\ &= e^{ik_0 x} e^{-i\omega(k_0)t} \int_{-\infty}^{+\infty} e^{-ak'^2} e^{ik'x} e^{-ik'v_g t} e^{-i\frac{1}{2}\beta k'^2 t} dk' \\ &= e^{i(k_0 x - \omega(k_0)t)} \int_{-\infty}^{+\infty} e^{-ak'^2} e^{i(k'(x - v_g t))} e^{-i\frac{1}{2}\beta k'^2 t} dk'\end{aligned}$$

$$= e^{i[k_0 x - \omega(k_0)t]} \int_{-\infty}^{\infty} e^{-\left(a + \frac{i\beta t}{2}\right)k'^2} e^{ik'x} dk'$$

Thus eqⁿ looks like eqⁿ (2) with
 $a \rightarrow a + \frac{i\beta t}{2}$ & $x \rightarrow x - v_g t$

$$\therefore \psi(x, t) = e^{i[k_0 x - \omega(k_0)t]} \sqrt{\frac{\pi}{a + \frac{i\beta t}{2}}} e^{-\frac{(x - v_g t)^2}{4(a + \frac{i\beta t}{2})}}$$

$$|\psi(x, t)|^2 = \sqrt{\frac{\pi^2}{a^2 + \frac{\beta^2 t^2}{4}}} e^{-\frac{(x - v_g t)^2}{2(a^2 + \frac{\beta^2 t^2}{4})}}$$

Eqⁿ (4) rep. a propagating wave packet with its peak moving with velocity v_g . Comp. eqⁿ (4) with (2A) we notice that quantity a at $t=0$ is rep. by $[a^2 + \frac{\beta^2 t^2}{4}]^{1/2}$ at time t . So the width of wavepacket grows larger as it propagates i.e. it spreads as it propagates.

Q. Find $\phi(k)$ for square wavepacket
 $\phi_0(x) = \begin{cases} A e^{ik_0 x} & |x| \leq a \\ 0 & |x| > a \end{cases}$